

Multiresolution Analysis for Compactly Supported Interpolating Tensor Product Wavelets (long version)

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Abstract

We construct a multidimensional interpolating tensor product MRA's of the function spaces $C_0(\mathbb{R}^n, K)$, $K = \mathbb{R}$ or $K = \mathbb{C}$, consisting of real or complex valued functions on \mathbb{R}^n vanishing at infinity and the function spaces $C_u(\mathbb{R}^n, K)$ consisting of bounded and uniformly continuous functions on \mathbb{R}^n . We also construct an interpolating dual MRA for both of these spaces. The theory of the tensor products of Banach spaces is used. We generalize the Besov space norm equivalence result from Donoho (1992, Interpolating Wavelet Transforms) to our n -dimensional construction.

Keywords: interpolating wavelets, multivariate wavelets, multiresolution analysis, tensor product, injective tensor norm, projective tensor norm, Besov space

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1 Introduction

Chui and Li [7] have constructed a one-dimensional MRA of function space $C_u(\mathbb{R})$ (bounded and uniformly continuous complex valued functions on \mathbb{R}) for interpolating wavelets. Donoho [16] has derived convergence results for interpolating wavelets on space $C_0(\mathbb{R})$. Goedecker's book [19] contains also some material about multidimensional interpolating wavelets. Goedecker [19] uses the term interpolating wavelets to mean Deslauriers-Dubuc wavelets whereas some other authors such as Chui and Li [7] use the term to mean roughly wavelet families whose mother scaling function has the cardinal interpolation property $\varphi(k) = \delta_{k,0}$ for all $k \in \mathbb{Z}$ (one-dimensional case). We follow the latter convention in this article. Chui and Li [6] have also constructed a general framework for multivariate wavelets. However, the theory in that article uses function space $L^2(\mathbb{R}^n)$ and so it is unsuitable for the approach in this article. Dubuc and Deslauriers [17, 11] have investigated interpolation processes related to the Deslauriers-Dubuc functions. Han and Jia [23] discuss fundamental functions (see the article for a definition) satisfying the cardinal interpolation property. Theory for orthonormal wavelets has been developed e.g. by Daubechies [8], Meyer [34], and Wojtaszczyk [42]. Kovačević and Sweldens [27] have investigated the use of digital filters for multidimensional biorthogonal wavelets. Numerical values for the wavelet filters of the Deslauriers-Dubuc wavelets have been given in [19] and [27]. Wavelets are also discussed in [24].

Reinov [37] has investigated Banach spaces without the approximation property. Brodzki and Niblo [5] have done some research on the rapid decay property of discrete groups and the metric approximation property.

The theory about tensor products of Banach spaces is presented e.g. in [38]. We use the notation from [38] for Banach space tensor products in this article. Schaefer's book [39] contains some material about tensor products of topological vector spaces. Book [33] contains also an introduction to tensor products of Banach spaces. Domański et al. [14, 15] have done some research on them. Michor [35]

represents tensor products of Banach spaces using category theory. Grothendieck [22] gives a classical presentation for tensor products of locally convex spaces. Theory for tensor products of Banach spaces can also be found in book [10]. Kustermans and Vaes [29] have developed some theory for space $C_0(G)$ where G is a compact or a locally compact group and for the tensor products of $C_0(G)$. Daws [9] presents some material on tensor products of Banach algebras. Glöckner [18] has shown that the tensor products of topological vector spaces are not associative.

Lewis [32] has constructed a MRA of function space $L^2(\mathbb{R})$ for interpolating wavelets in one dimension. DeVore et al. [12] have developed some theory for multidimensional orthonormal MRA of space $L^p(\mathbb{R}^d)$, $0 < p < \infty$, $d \in \mathbb{Z}_+$. He and Lai [25] have constructed nonseparable box spline wavelets on Sobolev spaces $H^s(\mathbb{R}^2)$.

An introduction to Besov spaces can be found e.g. in [4]. An introduction to Besov spaces using the Fourier transform based definition of these spaces can be found in [41]. DeVore and Popov [13] have investigated the connection of Besov spaces with the dyadic spline approximation and interpolation of Besov spaces. Kyriazis and Petrushev [30] give a method for the construction of unconditional bases for Triebel-Lizorkin and Besov spaces. The relationship between orthonormal wavelets and Besov spaces has been discussed by Meyer [34]. Almeida [1] has investigated wavelet bases in (generalized) Besov spaces.

Section 2 introduces some definitions used in the rest of this article. Some results on function spaces and sequences that are needed in the construction of the MRA's are given in section 3. Section 4 contains results on Banach space tensor products that are needed in the construction of the MRA's. We give some general definitions needed by the MRA's in section 5. A multivariate MRA of $C_u(\mathbb{R}^n, K)$, $K = \mathbb{R}$ or $K = \mathbb{C}$, is constructed in section 6. A multivariate MRA of $C_0(\mathbb{R}^n, K)$, $K = \mathbb{R}$ or $K = \mathbb{C}$, is constructed in section 7. The interpolating dual MRA is presented in section 8. The relationship between MRA's and infinite direct sums of Banach spaces is discussed in section 9. The Besov space norm equivalence from Donoho [16] is generalized for the n -dimensional interpolating MRA's in section 10.

2 Preliminaries

2.1 General

When $n \in \mathbb{N}$, $n \geq 2$, and P_1, \dots, P_n are propositions we define

$$P_1 \implies P_2 \implies \dots \implies P_n$$

to mean

$$(P_1 \implies P_2) \wedge (P_2 \implies P_3) \wedge \dots \wedge (P_{n-1} \implies P_n).$$

The set of all positive real numbers is denoted by \mathbb{R}_+ and the extended real line by \mathbb{R}_* . We define $\mathbb{R}_0 := \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}_{0*} := \{x \in \mathbb{R}_* : x \geq 0\}$, and \mathbb{Z}_+ to be the set of positive integer numbers. If A is a set and $P(x)$, $x \in A$, is a proposition then we define $\exists_{\text{def}} x \in A : P(x)$ to mean that $\exists y \in A : P(y)$ and x is defined to be some element of A for which $P(x)$ is true. When A and B are arbitrary sets, f is a function from A into B , and $X \subset A$ the image of X under f is denoted by $f[X]$. The set-theoretic support of a function $f : X \rightarrow \mathbb{C}$ where X is a set is denoted by $\text{supp}_{\text{set}} f$. The topological support of a function $f : T \rightarrow \mathbb{C}$ where T is a topological space is denoted by $\text{supp} f$. When E is a metric space, $x \in E$, and $r \in \mathbb{R}_+$ the closed ball of radius r centred at x is denoted by $\overline{B}_E(x; r)$. When A and B are some algebraic structures we may write $A \subset_{\text{set}} B$ to mean that B contains A as a subset. When A is a set we denote the cardinality of A by $\#A$.

Definition 2.1. When A is an arbitrary set define $\text{id}_A : A \rightarrow A$ to be the identity function on A , i.e. $\text{id}_A(x) = x$ for all $x \in A$.

Definition 2.2. When $n \in \mathbb{Z}_+$ define

$$Z(n) := \{k \in \mathbb{Z}_+ : k \leq n\}.$$

When $n \in \mathbb{N}$ define

$$Z_0(n) := \{k \in \mathbb{N} : k \leq n\}$$

and

$$Z_{\pm}(n) := \{k \in \mathbb{Z} : |k| \leq n\}.$$

Definition 2.3. When A is a set in which a commutative binary operation $+$ is defined, $B \subset A$, and $a \in A$ define

$$a + B := B + a := \{a + x : x \in B\}.$$

We define the differences $\Delta_{\mathbf{h}}^m$ and $\Delta_{\mathbf{h}}$ as in [41] and [3, def. V.4.1]:

$$(\Delta_{\mathbf{h}}^1 f)(\mathbf{x}) := (\Delta_{\mathbf{h}} f)(\mathbf{x}) := f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{h} \in \mathbb{R}^n$, and $f \in \mathbb{C}^{\mathbb{R}^n}$ and

$$\Delta_{\mathbf{h}}^m f := \Delta_{\mathbf{h}}(\Delta_{\mathbf{h}}^{m-1} f)$$

for all $m \in \mathbb{N} + 2$, $\mathbf{h} \in \mathbb{R}^n$, and $f \in \mathbb{C}^{\mathbb{R}^n}$.

Definition 2.4. When $n \in \mathbb{Z}_+$ and $E \subset \mathbb{C}$ define

$$\text{Bor}(\mathbb{R}^n, E) := \{f \in E^{\mathbb{R}^n} : f \text{ is Borel measurable}\}.$$

2.2 Sequences and Cartesian Products

Definition 2.5. When $n \in \mathbb{Z}_+$ define

$$\mathbf{0}_n := (0)_{k \in Z(n)}$$

and

$$\mathbf{1}_n := (1)_{k \in Z(n)}.$$

Definition 2.6. When $n \in \mathbb{Z}_+$ define $J_+(n) := \{0, 1\}^n \setminus \{\mathbf{0}_n\}$.

Definition 2.7. When I is a countable nonempty set define

$$\mathbf{e}_k^I := (\delta_{j,k})_{j \in I}$$

for all $k \in I$.

Definition 2.8. Define

$$\begin{aligned} \mathbf{e}_k &:= \mathbf{e}_k^{\mathbb{N}} && \text{for all } k \in \mathbb{N} \\ \check{\mathbf{e}}_k &:= \mathbf{e}_k^{\mathbb{Z}} && \text{for all } k \in \mathbb{Z} \\ \check{\mathbf{e}}_{\mathbf{k}} &:= \mathbf{e}_{\mathbf{k}}^{\mathbb{Z}^n} && \text{for all } \mathbf{k} \in \mathbb{Z}^n. \end{aligned}$$

When $n \in \mathbb{Z}_+$ define

$$\mathbf{e}_k^{[n]} := \mathbf{e}_k^{Z(n)} \quad \text{for all } k \in Z(n).$$

Definition 2.9. Let $n, m \in \mathbb{Z}_+$, $n \geq m$. Define

$$s_{\text{proj}}(m, \mathbf{s}) := (\mathbf{s}[1], \dots, \mathbf{s}[m]) \in \mathbb{C}^m$$

where $\mathbf{s} \in \mathbb{C}^n$.

Definition 2.10. Let $n, m \in \mathbb{Z}_+$. Define

$$s_{\text{comb}}(\mathbf{s}, \mathbf{t}) := (\mathbf{s}[1], \dots, \mathbf{s}[n], \mathbf{t}[1], \dots, \mathbf{t}[m]) \in \mathbb{C}^{n+m}$$

where $\mathbf{s} \in \mathbb{C}^n$ and $\mathbf{t} \in \mathbb{C}^m$.

2.3 Vector Spaces

Suppose that A and B are topological vector spaces. We define $A =_{\text{tvs}} B$ to be true iff A and B are the same topological vector space, i.e. A and B are the same vector space and have the same topology. A and B are called *algebraically isomorphic* iff A and B are isomorphic as vector spaces. A function $\iota : A \rightarrow B$ is called an *algebraic isomorphism* iff ι is an isomorphism from vector space A onto vector space B (i.e. ι is a linear bijection from A onto B). A function $\eta : A \rightarrow B$ is called a *topological isomorphism* iff η is an algebraic isomorphism from A onto B and a homeomorphism from topological space A onto topological space B (i.e. ι is an algebraic isomorphism that preserves the topology). A and B are called *topologically isomorphic* iff there exists a topological isomorphism from A onto B . We define $\mathcal{L}(A, B)$ to be the set of all continuous linear functions from A into B .

When B is a topological vector space we define $A \subset_{\text{tvs}} B$ to be true iff A is a topological vector subspace of B .

Suppose that E and F are normed vector spaces. We define $E =_{\text{n.s.}} F$ to be true iff E and F are the same normed vector space, i.e. E and F are the same vector space (they consist of the same elements, have the same scalar field, and the addition and scalar multiplication operations are the same) and have the same norm. We define $E \cong_1 F$ to be true iff E and F are isometrically isomorphic.

The term *operator* shall mean a bounded linear function from a normed vector space into another one. The term *projection* shall mean a linear projection of a vector space onto a vector space. The topological dual of a Banach space A is denoted by A^* . Unless otherwise stated A^* is equipped with the norm topology. The unit ball of a normed vector space A is denoted by B_A . When A is a Banach space and $(\tilde{x}_k)_{k=0}^\infty$ is a sequence in A^* we use the notation $\tilde{x}_k \xrightarrow{w^*} \tilde{x}$ to mean that the sequence $(\tilde{x}_k)_{k=0}^\infty$ converges to \tilde{x} in the weak-* topology of A^* . Convergence in the norm topology is denoted by $\tilde{x}_k \rightarrow \tilde{x}$.

When E is a Banach space and A and B open or closed subspaces of E the topological direct sum of A and B is denoted by $A \dot{+} B$, i.e. $A \dot{+} B$ is the algebraic direct sum of vector spaces A and B and both A and B are topologically complemented in the normed vector space $A \dot{+} B$.

When A and B are Banach spaces we use the notation $A \subset_1 B$ to mean that A is isometrically embedded in B , i.e. $A \subset B$ and the inclusion map is distance preserving. When B is a Banach space we use the notation $A \subset_{\text{c.s.}} B$ to mean that A is a closed subspace of B . When E is a Banach space, $A \subset_{\text{set}} E$, and $B \subset_{\text{set}} E^*$ we define $B \perp A$ to be true iff $\langle \tilde{b}, a \rangle = 0$ for all $a \in A$ and $\tilde{b} \in B$.

$$\forall a \in A, \tilde{b} \in B : \langle \tilde{b}, a \rangle = 0.$$

When E is a normed vector space and $x \in E$ we may use the notation $\|x|E\|$ to mean the norm of x in E . When B is a normed vector space and we write $A :=_{\text{n.s.}} \{x \in B : P(x)\}$ we assume that $\|x|A\| := \|x|B\|$ for each $x \in A$. When A is a vector space we may define a norm $\|\cdot\|_A$ on A so that $\|\cdot\|_A$ is defined on a larger set E containing A as a subset.

Definition 2.11. Let E be a set and A be a vector space so that $A \subset_{\text{set}} E$. Suppose that $\|\cdot\|_A : E \rightarrow \mathbb{R}_{0+}$ is a norm on A . We say that the norm $\|\cdot\|_A$ characterizes A on E iff $\|x\|_A < \infty$ for all $x \in A$ and $\|y\|_A = \infty$ for all $y \in E \setminus A$.

2.4 Function Spaces and Sequence Spaces

When $p \in [1, \infty]$, B is a Banach space, and I a denumerable set we denote the l^p space consisting of a subset of B^I by $l^p(I, B)$. As usual, $l^p(I) :=_{\text{n.s.}} l^p(I, \mathbb{C})$. When $K = \mathbb{R}$ or $K = \mathbb{C}$, $n \in \mathbb{Z}_+$, and $p \in [1, \infty]$ we denote the L^p space consisting of a subset of the Borel measurable functions from \mathbb{R}^n into K by $L^p(\mathbb{R}^n, K)$. As usual, $L^p(\mathbb{R}^n) :=_{\text{n.s.}} L^p(\mathbb{R}^n, \mathbb{C})$.

Definition 2.12. Let T be a topological space and $K = \mathbb{R}$ or $K = \mathbb{C}$. Define

$$C_b(T, K) := \{f : f \text{ is a continuous and bounded function from } T \text{ to } K\}$$

where the norm of an element f is given by

$$\|f\| = \sup_{x \in T} |f(x)|$$

and $C_b(T) :=_{\text{n.s.}} C_b(T, \mathbb{C})$.

$C_b(T, K)$ is a Banach space.

When f is a function from \mathbb{R}^n into \mathbb{C} and $a \in \mathbb{C}$

$$\lim_{\|x\| \rightarrow \infty} f(x) = a$$

is equivalent to

$$\forall \varepsilon \in \mathbb{R}_+ : \exists r \in \mathbb{R}_+ : \forall \mathbf{x} \in \mathbb{R}^n \setminus \overline{B}_{\mathbb{R}^n}(0; r) : |f(\mathbf{x}) - a| < \varepsilon.$$

Definition 2.13. Let E be a metric space and $K = \mathbb{R}$ or $K = \mathbb{C}$. Define

$$C_u(E, K) :=_{\text{n.s.}} \{f \in C_b(E, K) : f \text{ is uniformly continuous on } E\}$$

and $C_u(E) :=_{\text{n.s.}} C_u(E, \mathbb{C})$.

Functions vanishing at infinity are defined as in [31].

Definition 2.14. Let T be a locally compact Hausdorff space and $K = \mathbb{R}$ or $K = \mathbb{C}$. Function $f \in C_b(T, K)$ is said to *vanish at infinity* if the set $\{t \in T : |f(t)| \geq \varepsilon\}$ is compact for all $\varepsilon \in \mathbb{R}_+$. The Banach space of all functions vanishing at infinity is defined by

$$C_0(T, K) :=_{\text{n.s.}} \{f \in C_b(T, K) : f \text{ vanishes at infinity}\}$$

and $C_0(T) :=_{\text{n.s.}} C_0(T, \mathbb{C})$.

Definition 2.15. Let T be a locally compact Hausdorff space and $K = \mathbb{R}$ or $K = \mathbb{C}$. Define

$$C_{\text{com}}(T, K) :=_{\text{n.s.}} \{f \in C_b(T, K) : \text{supp } f \text{ is compact}\}$$

and $C_{\text{com}}(T) :=_{\text{n.s.}} C_{\text{com}}(T, \mathbb{C})$.

We have $C_{\text{com}}(\mathbb{R}^n) \subset_{\text{n.s.}} C_0(\mathbb{R}^n) \subset_{\text{c.s.}} C_u(\mathbb{R}^n) \subset_{\text{c.s.}} C_b(\mathbb{R}^n)$ for each $n \in \mathbb{Z}_+$ [16, section 2], [31].

Definition 2.16. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let E be a closed subspace of $C_b(\mathbb{R}^n, K)$ for which the following condition is true:

$$\forall f \in C_b(\mathbb{R}^n, K), c \in \mathbb{R}_+, \mathbf{d} \in \mathbb{R}^n : f \in E \iff f(c \cdot -\mathbf{d}) \in E.$$

Let $a \in \mathbb{R}_+$, $\mathbf{b} \in \mathbb{R}^n$, and $\tilde{f} \in E^*$. The *a-dilatation and b-translation* of \tilde{f} , denoted by $\tilde{f}(a \cdot -\mathbf{b})$, is defined by [7]

$$\langle \tilde{f}(a \cdot -\mathbf{b}), f \rangle := \frac{1}{a^n} \left\langle \tilde{f}, f \left(\frac{\cdot + \mathbf{b}}{a} \right) \right\rangle$$

for all $f \in E$.

We define the modulus of continuity in the standard way [3, def. V.4.2]:

Definition 2.17. When $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}_+$, $p \in [1, \infty]$, and $t \in \mathbb{R}_0$ define

$$\begin{aligned} \omega_p^m(f; t) &:= \sup\{\|\Delta_{\mathbf{h}}^m f\|_{L^p(\mathbb{R}^n)} : \mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; t)\}, & f \in \text{Bor}(\mathbb{R}^n, \mathbb{C}), \\ \omega_\infty^m(f; t) &:= \sup\{\|\Delta_{\mathbf{h}}^m f\|_{C_b(\mathbb{R}^n)} : \mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; t)\}, & f \in C_b(\mathbb{R}^n), \end{aligned}$$

and

$$\omega(f; t) := \omega_\infty^1(f; t), \quad f \in C_b(\mathbb{R}^n).$$

We have

$$\lim_{t \rightarrow 0} \omega(f; t) = 0$$

for all $f \in C_u(\mathbb{R}^n)$ and $n \in \mathbb{Z}_+$.

The Besov spaces on \mathbb{R}^n are denoted by $B_{p,q}^s(\mathbb{R}^n)$. There are two ways to define Besov spaces and for certain combinations of parameters these yield different spaces. One way to define these spaces is based on the Fourier transform and the other on the modulus of continuity. The former definition is used e.g. by Peetre [36] and the latter by DeVore and Popov [13]. Triebel [41] uses the term ‘‘Besov space’’ for the spaces defined by the latter definition and denotes them by $\Lambda_{p,q}^s$ and he denotes the spaces defined by the former definition by $B_{p,q}^s$. Both definitions yield the same spaces for the range of parameters $s > \frac{n}{p}$ used in this article [16]. The two definitions also yield the same spaces $B_{p,q}^s(\mathbb{R}^n)$ for $s > 0$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$ [41]. The spaces $B_{p,q}^s(\mathbb{R}^n)$ are quasi-Banach spaces for $-\infty < s < \infty$, $0 < p \leq \infty$, $0 < q \leq \infty$ and Banach spaces if $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ [41].

We have $\mathcal{Z}^s(\mathbb{R}^n) =_{\text{tvs}} B_{\infty,\infty}^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}_+$ where $\mathcal{Z}^s(\mathbb{R}^n)$ are the Hölder-Zygmund spaces [41]. We also have $C^s(\mathbb{R}^n) =_{\text{tvs}} \mathcal{Z}^s(\mathbb{R}^n) =_{\text{tvs}} B_{\infty,\infty}^s(\mathbb{R}^n)$ when $s \in \mathbb{R}_+ \setminus \mathbb{Z}_+$ where $C^s(\mathbb{R}^n)$, $s \in \mathbb{R}_+ \setminus \mathbb{Z}_+$ are the Hölder spaces on \mathbb{R}^n [41]. When $m \in \mathbb{N}$ we denote the Banach space of functions with bounded and uniformly continuous partial derivatives up to m th degree equipped with the usual norm by $C^m(\mathbb{R}^n)$, see [41]. When $m \in \mathbb{Z}_+$ we have $C^m(\mathbb{R}^n) \subset_{\text{set}} \mathcal{Z}^m(\mathbb{R}^n)$.

Definition 2.18. Let $n \in \mathbb{R}_+$, $s \in \mathbb{R}_+$, $p \in [1, \infty]$, and $q \in [1, \infty]$. Let $m \in \mathbb{Z}_+$ and $m > s$. When $p < \infty$ define

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n);m}^{(c)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\| t \in]0, 1[\mapsto t^{-s-\frac{1}{q}} \omega_p^m(f; t) \right\|_{L^q(]0,1[)}$$

for all $f \in \text{Bor}(\mathbb{R}^n, \mathbb{C})$. When $p = \infty$ define

$$\|f\|_{B_{\infty,q}^s(\mathbb{R}^n);m}^{(c)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \left\| t \in]0, 1[\mapsto t^{-s-\frac{1}{q}} \omega_\infty^m(f; t) \right\|_{L^q(]0,1[)}$$

for all $f \in C_b(\mathbb{R}^n)$.

Norm $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^n);m}^{(c)}$ is an equivalent norm on $B_{p,q}^s(\mathbb{R}^n)$ for the range of parameters given in Definition 2.18. Replacing the ranges $]0, 1[$ by $]0, \infty[$ in the definition above results equivalent norms [3, def. V.4.3]. Denote these norms by $\|f\|_{B_{p,q}^s(\mathbb{R}^n);m}^{(c')}$. Norms $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^n);m}^{(c)}$ and $\|\cdot\|_{B_{p,q}^s(\mathbb{R}^n);m}^{(c')}$ characterize the Besov space $B_{p,q}^s(\mathbb{R}^n)$ on $\text{Bor}(\mathbb{R}^n, \mathbb{C})$ when $s \in \mathbb{R}_+$, $p \in [1, \infty[$, and $q \in [1, \infty]$. Norms $\|\cdot\|_{B_{\infty,q}^s(\mathbb{R}^n);m}^{(c)}$ and $\|\cdot\|_{B_{\infty,q}^s(\mathbb{R}^n);m}^{(c')}$ characterize the Besov space $B_{\infty,q}^s(\mathbb{R}^n)$ on $C_b(\mathbb{R}^n)$ when $s \in \mathbb{R}_+$ and $q \in [1, \infty]$. [3, def. V.4.3].

2.5 Tensor Products

Definition 2.19. When $n \in \mathbb{Z}_+$ define

$$\mathbf{e}_k^\otimes := \bigotimes_{l=1}^n \mathbf{e}_{k[l]}$$

and

$$\check{\mathbf{e}}_k^\otimes := \bigotimes_{l=1}^n \check{\mathbf{e}}_{k[l]}.$$

The square ordering is defined as in [38] and [40] in the following.

Definition 2.20. Define function $\sigma_{\text{sq}} : \mathbb{N} \rightarrow \mathbb{N}^2$ by

$$\sigma_{\text{sq}}(0) = (0, 0) \quad (1)$$

and

$$\sigma_{\text{sq}}(k) = \begin{cases} (i, n); & k = n^2 + i \wedge i \in Z_0(n) \wedge n \in \mathbb{N} \\ (n, n - i); & k = n^2 + n + i \wedge i \in Z(n) \wedge n \in \mathbb{N} \end{cases} \quad (2)$$

Function σ_{sq} is called the square ordering. Define also functions $\sigma_{\text{sq}1} : \mathbb{N} \rightarrow \mathbb{N}$ and $\sigma_{\text{sq}2} : \mathbb{N} \rightarrow \mathbb{N}$ by setting $\sigma_{\text{sq}1}(k) := \sigma_{\text{sq}}(k)[1]$ and $\sigma_{\text{sq}2}(k) := \sigma_{\text{sq}}(k)[2]$ for all $k \in \mathbb{N}$.

Function σ_{sq} is a bijection from \mathbb{N} onto \mathbb{N}^2 . The tensor product basis is defined as in [38]:

Definition 2.21. Let E and F be Banach spaces that both have Schauder bases. Let $(a_k)_{k=0}^\infty$ and $(b_k)_{k=0}^\infty$ be Schauder bases of E and F , respectively. The sequence $(a_{\sigma_{\text{sq}1}(k)} \otimes b_{\sigma_{\text{sq}2}(k)})_{k=0}^\infty \subset E \otimes F$ is called the *tensor product basis generated by Schauder bases* $(a_k)_{k=0}^\infty$ and $(b_k)_{k=0}^\infty$.

The tensor product basis is a Schauder basis for both $E \hat{\otimes}_\varepsilon F$ and $E \hat{\otimes}_\pi F$ [38, 40].

When E and F are Banach spaces consisting of complex valued functions (in particular, real valued functions), $u \in E$, and $v \in F$ the tensor product $u \otimes v$ is identified with a function on the Cartesian product of the domains of u and v by setting $(u \otimes v)(x \otimes y) = u(x)v(y)$ where x belongs to the domain of u and y belongs to the domain of v .

When $n \in \mathbb{Z}_+$, α is a uniform crossnorm, and A_1, \dots, A_n are Banach spaces then the completed tensor product of several Banach spaces is defined by the recursive formula

$$A_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha A_n :=_{\text{n.s.}} (A_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha A_{n-1}) \hat{\otimes}_\alpha A_n.$$

The indexed completed tensor product is defined by

$$\bigotimes_{j=1}^n \hat{\otimes}_\alpha A_j :=_{\text{n.s.}} \left(\bigotimes_{j=1}^{n-1} \hat{\otimes}_\alpha A_j \right) \hat{\otimes}_\alpha A_n$$

where $n > 1$. When $n = 1$ define

$$\bigotimes_{j=1}^1 \hat{\otimes}_\alpha A_j :=_{\text{n.s.}} A_1.$$

The concept of norm in tensor product spaces is used in two slightly different ways. When we speak about norm α defined on space $E \otimes F$ we mean that α is a function that maps each element $u \in E \otimes F$ to its norm $\|u\|$ in \mathbb{R}_0 . On the other hand uniform crossnorms and tensor norms are assignments of a reasonable crossnorm to each pair of Banach spaces.

A tensor norm is defined to be a finitely generated uniform crossnorm [38]. The Schatten dual of a tensor norm α is denoted by α^s . When α is a tensor norm α^s is a uniform crossnorm. Uniform crossnorms do not generally respect subspaces. I.e. if X is a closed subspace of a Banach space E and Y is a closed subspace of a Banach space F the norm of an element $u \in X \hat{\otimes}_\alpha Y$ is not necessary equal in spaces $X \hat{\otimes}_\alpha Y$ and $E \hat{\otimes}_\alpha F$ [38]. However, it is possible to use the norm inherited from $E \hat{\otimes}_\alpha F$ in vector space $X \otimes Y$. Then the closure of $X \otimes Y$ with the inherited norm in $E \hat{\otimes}_\alpha F$ is a closed subspace of $E \hat{\otimes}_\alpha F$ [33, chapter 1]. We give the following definition for this kind of construction.

Definition 2.22. Let E and F be Banach spaces and α a norm on $E \otimes F$. Let X be a closed subspace of E and Y a closed subspace of F . Define $X \otimes_{\alpha; E \hat{\otimes}_\alpha F} Y$ to be the normed vector space formed by using the norm inherited from $E \hat{\otimes}_\alpha F$ in the vector space $X \otimes Y$, i.e. $\|u\|_{X \otimes_{\alpha; E \hat{\otimes}_\alpha F} Y} := \alpha_{E, F}(u)$ for all $u \in X \otimes Y$. Define $X \hat{\otimes}_{\alpha; E \hat{\otimes}_\alpha F} Y$ to be the closure of $X \otimes_{\alpha; E \hat{\otimes}_\alpha F} Y$ in space $E \hat{\otimes}_\alpha F$. The notations $X \otimes_{\alpha; E \otimes F} Y$ and $X \hat{\otimes}_{\alpha; E \otimes F} Y$ may also be used for the aforementioned definitions.

Now $X \otimes_{\alpha; E \hat{\otimes}_{\alpha} F} Y$ is a normed subspace of $E \hat{\otimes}_{\alpha} F$ and $X \hat{\otimes}_{\alpha; E \hat{\otimes}_{\alpha} F} Y$ is a closed subspace of $E \hat{\otimes}_{\alpha} F$.

If α is a reasonable crossnorm, $a \in X^*$, and $b \in Y^*$ then $a \otimes b$ is a continuous linear functional on $X \otimes_{\alpha} Y$ and $\|a \otimes b\| = \|a\| \|b\|$ [38]. We have

$$(a \otimes b)(x \otimes y) = a(x)b(y) \quad (3)$$

for all $x \in X$ and $y \in Y$ and the elements of $X^* \otimes Y^*$ may be interpreted as linear functionals on $X \otimes_{\alpha} Y$. There is an isometric embedding

$$X^* \otimes_{\alpha^s} Y^* \subset_1 (X \otimes_{\alpha} Y)^*. \quad (4)$$

for tensor norms α [38]. See also [10].

3 Some Theorems on Function Spaces and Sequences

Lemma 3.1. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let I be a denumerable set and $\mathbf{a} \in K^I$. Suppose that σ_1 and σ_2 are bijections from \mathbb{N} onto I . Then*

$$(\mathbf{a}[\sigma_1(k)])_{k=0}^{\infty} \in c_0(\mathbb{N}, K) \iff (\mathbf{a}[\sigma_2(k)])_{k=0}^{\infty} \in c_0(\mathbb{N}, K).$$

Proof. Suppose that $(\mathbf{a}[\sigma_1(k)])_{k=0}^{\infty} \in c_0(\mathbb{N}, K)$. Let $h > 0$. Then there exists $n_1 \in \mathbb{N}$ so that $|\mathbf{a}[\sigma_1(k)]| < h$ for each $k \in \mathbb{N}$, $k > n_1$. Let

$$n_2 := \max \{ \sigma_2^{-1}(\sigma_1(k)) : k \in Z_0(n_1) \}.$$

Suppose that $k \in \mathbb{N}$, $k > n_2$. Then

$$\begin{aligned} k &\neq \sigma_2^{-1}(\sigma_1(l)) \quad \forall l \in Z_0(n_1) \\ \implies \sigma_2(k) &\neq \sigma_1(k) \quad \forall l \in Z_0(n_1) \\ \implies \sigma_2(k) &= \sigma_1(l_0) \text{ for some } l_0 \in \mathbb{N}, l_0 > n_1 \\ \implies |\mathbf{a}[\sigma_2(k)]| &= |\mathbf{a}[\sigma_1(l_0)]| < h. \end{aligned}$$

Hence

$$\lim_{k \rightarrow \infty} \mathbf{a}[\sigma_2(k)] = 0.$$

and so $(\mathbf{a}[\sigma_2(k)])_{k=0}^{\infty} \in c_0(\mathbb{N}, K)$. Similarly, if $(\mathbf{a}[\sigma_2(k)])_{k=0}^{\infty} \in c_0(\mathbb{N}, K)$ it follows that $(\mathbf{a}[\sigma_1(k)])_{k=0}^{\infty} \in c_0(\mathbb{N}, K)$. \square

Consequently the following definition makes sense.

Definition 3.2. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and I be a denumerable set. Define

$$\begin{aligned} c_0(I, K) &:= \{ (a_{\lambda})_{\lambda \in I} : (a_{\sigma(k)})_{k=0}^{\infty} \in c_0(\mathbb{N}, K) \text{ for some bijection } \sigma : \mathbb{N} \rightarrow I \} \\ \|\mathbf{a}\|_{c_0(I, K)} &:= \|\mathbf{a}\|_{\infty}, \quad \mathbf{a} \in c_0(I, K). \end{aligned}$$

Banach space $c_0(I, K)$ is isometrically isomorphic to $c_0(\mathbb{N}, K)$.

Definition 3.3. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let f be a function from \mathbb{R}^n into K . Define

$$N_{\text{cover}}(f) := \max_{\mathbf{x} \in \mathbb{R}^n} \# \{ \mathbf{k} \in \mathbb{Z}^n : f(\mathbf{x} - \mathbf{k}) \neq 0 \}.$$

$N_{\text{cover}}(f) < \infty$ for all compactly supported functions $f : \mathbb{R}^n \rightarrow K$. When $a \in \mathbb{R}_+$ function $\mathbf{y} \in \mathbb{R}^n \mapsto a\mathbf{y}$ is a bijection from \mathbb{R}^n onto \mathbb{R}^n and it follows that

$$\max_{\mathbf{x} \in \mathbb{R}^n} \# \{ \mathbf{k} \in \mathbb{Z}^n : f(a\mathbf{x} - \mathbf{k}) \neq 0 \} = N_{\text{cover}}(f). \quad (5)$$

Definition 3.4. Let E and F be normed vector spaces. When f is a compactly supported function from E into F define

$$r_{\text{supp}}(f) := \inf\{r \in \mathbb{R}_0 : \text{supp}_{\text{set}} f \subset \overline{B}_E(0; r)\}.$$

Definition 3.5. Let $n \in \mathbb{Z}_+$. Define

$$I_{\text{trans}}(f, \mathbf{x}) := \{\mathbf{k} \in \mathbb{Z}^n : f(\mathbf{x} - \mathbf{k}) \neq 0\}$$

for all $f \in \mathbb{C}^{\mathbb{R}^n}$ and $\mathbf{x} \in \mathbb{R}^n$.

Definition 3.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $n \in \mathbb{Z}_+$. Define

$$I_{\text{rect}}(\mathbf{a}, \mathbf{b}) = \prod_{k=1}^n I_1(\mathbf{a}[k], \mathbf{b}[k])$$

where

$$I_1(u, v) = [\min\{u, v\}, \max\{u, v\}] \cap \mathbb{Z}$$

for all u, v in \mathbb{R} .

Lemma 3.7. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $c \in \mathbb{R}_+$, and $f \in C_{\text{com}}(\mathbb{R}^n, K)$. When $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$ series

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k})$$

converges unconditionally in $C_0(\mathbb{R}^n, K)$ and

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k}) \right\|_{\infty} \leq N_{\text{cover}}(f) \|f\|_{\infty} \|\mathbf{a}\|_{\infty}. \quad (6)$$

Proof. Let $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. Let $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection. Let

$$s_l = \sum_{i=0}^l \mathbf{a}[\sigma(i)] f(c \cdot -\sigma(i))$$

Let $l, l' \in \mathbb{N}$, $l' > l$. Now

$$s_{l'} - s_l = \sum_{i=l+1}^{l'} \mathbf{a}[\sigma(i)] f(c \cdot -\sigma(i)).$$

It follows from Equation (5) that

$$|s_{l'}(\mathbf{x}) - s_l(\mathbf{x})| = \left| \sum_{i=l+1}^{l'} \mathbf{a}[\sigma(i)] f(c\mathbf{x} - \sigma(i)) \right| \leq N_{\text{cover}}(f) b_{l,l'} m$$

where $b_{l,l'} = \max\{|\mathbf{a}[\sigma(i)]| : i = l+1, \dots, l'\}$ and $m = N_{\text{cover}}(f) \|f\|_{\infty}$. Let $h > 0$. There exists $i_0 \in \mathbb{N}$ so that

$$|\mathbf{a}[\sigma(i)]| < \frac{h}{m}$$

for all $i \in \mathbb{N}$ and $i > i_0$. Now $b_{l,l'} < \frac{h}{m}$ for all $\forall l, l' > i_0$ and $l' > l$. We also have

$$|s_{l'}(\mathbf{x}) - s_l(\mathbf{x})| < \frac{h}{m} \cdot m = h$$

for all $\mathbf{x} \in \mathbb{R}$ and for all $l, l' \in \mathbb{N}$, $l, l' > i_0$, $l' > l$. Hence $(s_l)_{l=0}^\infty$ is a Cauchy sequence in $C_0(\mathbb{R}^n, K)$ and it converges in $C_0(\mathbb{R}^n, K)$. Therefore

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k})$$

converges unconditionally in $C_0(\mathbb{R}^n, K)$. It follows from Equation (5) that

$$\left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c\mathbf{x} - \mathbf{k}) \right| \leq N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty$$

for all $\mathbf{x} \in \mathbb{R}^n$. Hence inequality (6) is true. \square

Lemma 3.8. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $c \in \mathbb{R}_+$. Let $f \in C_{\text{com}}(\mathbb{R}^n, K)$ so that*

$$f(\mathbf{k} + \mathbf{b}) = \delta_{\mathbf{k}, 0} \quad (7)$$

for all $\mathbf{k} \in \mathbb{Z}^n$ for some constant $\mathbf{b} \in \mathbb{R}^n$. Then

$$\|\mathbf{a}\|_\infty \leq \left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k}) \right\|_\infty \leq N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty \quad (8)$$

for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$.

Proof. Let $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. By Lemma 3.7 series

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k})$$

converges unconditionally in $C_0(\mathbb{R}^n, K)$ and the right-hand side of inequality (8) is true. Let $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection and $m := \|\mathbf{a}\|_\infty$. Since

$$\lim_{l \rightarrow \infty} \mathbf{a}[\sigma(l)] = 0$$

there exists $l_0 \in \mathbb{N}$ so that $|\mathbf{a}[\sigma(l_0)]| = m$. Now

$$\begin{aligned} \left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k}) \right\|_\infty &\geq \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f\left(c \frac{\sigma(l_0) + \mathbf{b}}{c} - \mathbf{k}\right) \right| \\ &= \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(\sigma(l_0) + \mathbf{b} - \mathbf{k}) \right| \\ &= |\mathbf{a}[\sigma(l_0)]| = \|\mathbf{a}\|_\infty. \end{aligned}$$

\square

It follows from Equation (7) that $\|f\|_\infty \geq 1$ and $N_{\text{cover}}(f) \geq 1$ under the conditions of Lemma 3.8.

Lemma 3.9. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $c \in \mathbb{R}_+$. Let $f \in C_{\text{com}}(\mathbb{R}^n, K)$ so that $f(\mathbf{k} + \mathbf{b}) = \delta_{\mathbf{k}, 0}$ for all $\mathbf{k} \in \mathbb{Z}^n$ for some $\mathbf{b} \in \mathbb{R}^n$. Let*

$$\begin{aligned} V &:= \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k}) : \mathbf{a} \in c_0(\mathbb{Z}^n, K) \right\} \\ \|g|V\| &:= \|g\|_\infty, \quad g \in V. \end{aligned} \quad (9)$$

Define function $\iota : c_0(\mathbb{Z}^n, K) \rightarrow V$ by

$$\iota(\mathbf{a}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k})$$

for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. Then

(i) V is a closed subspace of $C_0(\mathbb{R}^n, K)$.

(ii) Function ι is a topological isomorphism from $c_0(\mathbb{Z}^n, K)$ onto V and

$$\|\mathbf{a}\|_\infty \leq \|\iota(\mathbf{a})\|_\infty \leq N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty \quad (10)$$

for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$.

Proof. The series in Equation (9) converges unconditionally by Lemma 3.7. Function ι is linear and a surjection onto V . By Lemma 3.8 inequality (10) is true.

By the definition of ι we have $\|\mathbf{a}\|_\infty \leq \|\iota(\mathbf{a})\|_\infty$ for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. Suppose that $\mathbf{w}, \mathbf{z} \in c_0(\mathbb{Z}^n, K)$ and $\mathbf{w} \neq \mathbf{z}$. Now $\|\iota(\mathbf{w} - \mathbf{z})\|_\infty \geq \|\mathbf{w} - \mathbf{z}\|_\infty > 0$. Hence ι is an injection. It follows that ι is a linear bijection and vector spaces V and $c_0(\mathbb{Z}^n, K)$ are algebraically isomorphic. We have $\|\mathbf{a}\|_\infty \leq \|\iota(\mathbf{a})\|_\infty \leq N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty$ for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. Hence ι and ι^{-1} are continuous. Thus (ii) is true.

Since V and $c_0(\mathbb{Z}^n, K)$ are topologically isomorphic and $c_0(\mathbb{Z}^n, K)$ is a Banach space it follows that V is also a Banach space and consequently a closed subspace of $C_0(\mathbb{R}^n, K)$. Thus (i) is true. \square

Lemma 3.10. Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $c \in \mathbb{R}_+$. Let $f \in C_{\text{com}}(\mathbb{R}^n, K)$ and $f(\mathbf{k} + \mathbf{b}) = \delta_{\mathbf{k}, 0}$ for all $\mathbf{k} \in \mathbb{Z}^n$ for some $\mathbf{b} \in \mathbb{R}^n$. Let

$$V := \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c\mathbf{x} - \mathbf{k}) : \mathbf{a} \in l^\infty(\mathbb{Z}^n, K) \right\}$$

$$\|g|V\| := \|g\|_\infty.$$

Define function $\iota : l^\infty(\mathbb{Z}^n, K) \rightarrow V$ by

$$\iota(\mathbf{a}) := \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c\mathbf{x} - \mathbf{k})$$

for all $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$. Then

(i) V is a closed subspace of $C_u(\mathbb{R}^n, K)$.

(ii) Function ι is a topological isomorphism from $l^\infty(\mathbb{Z}^n, K)$ onto V and

$$\|\mathbf{a}\|_\infty \leq \|\iota(\mathbf{a})\|_\infty \leq N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty$$

for all $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$.

Lemma 3.11. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $E =_{\text{n.s.}} C_u(\mathbb{R}^n, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. Let $c \in \mathbb{R}_+$, $\tilde{f} \in E^*$, and $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$. Then the series

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k})$$

converges absolutely in E^* and

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k}) \right\|_{E^*} \leq \frac{\|\tilde{f}\|}{c^n} \|\mathbf{d}\|_1$$

Proof. Now

$$\left\| \tilde{f}(c \cdot -\mathbf{k}) \right\| = \frac{1}{c^n} \left\| \tilde{f} \right\|$$

for all $\mathbf{k} \in \mathbb{Z}^n$ and

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \left\| \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k}) \right\| = \frac{\left\| \tilde{f} \right\|}{c^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} |\mathbf{d}[\mathbf{k}]|$$

that converges because $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$. Consequently

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k}) \right\| \leq \sum_{\mathbf{k} \in \mathbb{Z}^n} \left\| \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k}) \right\| = \frac{\left\| \tilde{f} \right\|}{c^n} \|\mathbf{d}\|_1.$$

□

Absolute convergence of a sequence in a Banach space implies unconditional convergence.

Lemma 3.12. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $E =_{\text{n.s.}} C_u(\mathbb{R}^n, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. Let $c \in \mathbb{R}_+$. Then*

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \delta(c \cdot -\mathbf{k}) \Big| E^* \right\| = \frac{1}{c^n} \|\mathbf{d}\|_1$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$.

Proof. See also the proof of [7, theorem 2.6]. By Lemma 3.11

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \delta(c \cdot -\mathbf{k}) \Big| E^* \right\| \leq \frac{1}{c^n} \|\mathbf{d}\|_1$$

and the series on the left-hand side converges absolutely. Let

$$h(x) := \begin{cases} 4x + 1; & \text{if } x \in [-\frac{1}{4}, 0[\\ -4x + 1; & \text{if } x \in [0, \frac{1}{4}[\\ 0; & \text{otherwise} \end{cases}$$

where $x \in \mathbb{R}$ and $h^{[n]} : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$h^{[n]} = \bigotimes_{k=1}^n h.$$

Let $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection. Suppose that $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. By Lemma 3.7 series $\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] h(c \cdot -\mathbf{k})$ converges unconditionally in $C_0(\mathbb{R}^n, K)$. By Lemma 3.11 the series $\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \delta(c \cdot -\mathbf{k})$ converges absolutely. Furthermore,

$$\langle \delta(c \cdot -\ell), h^{[n]}(c \cdot -\mathbf{k}) \rangle = \frac{1}{c^n} \left\langle \delta \left(\cdot - \frac{\ell}{c} \right), h^{[n]}(c \cdot -\mathbf{k}) \right\rangle = \delta_{\ell, \mathbf{k}}$$

for all $\mathbf{k}, \ell \in \mathbb{Z}^n$ and

$$\left\langle \sum_{\ell \in \mathbb{Z}^n} \mathbf{d}[\ell] \delta(c \cdot -\ell), \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] h^{[n]}(c \cdot -\mathbf{k}) \right\rangle = \frac{1}{c^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}[\mathbf{k}]$$

and

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] h^{[n]}(c \cdot -\mathbf{k}) \right\|_{\infty} = \|\mathbf{a}\|_{\infty}.$$

So

$$\frac{1}{c^n} \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}[\mathbf{k}] \right| = \left| \left\langle \sum_{\ell \in \mathbb{Z}^n} \mathbf{d}[\ell] \delta(c \cdot -\ell), \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] h^{[n]}(c \cdot -\mathbf{k}) \right\rangle \right| \leq \left\| \sum_{\ell \in \mathbb{Z}^n} \mathbf{d}[\ell] \delta(c \cdot -\ell) \right\|_{E^*} \|\mathbf{a}\|_{\infty}.$$

Hence

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \delta(c \cdot -\mathbf{k}) \right\|_{E^*} \geq \frac{1}{c^n} \sup \left\{ \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}[\mathbf{k}] \right| : \mathbf{a} \in c_0(\mathbb{Z}^n, K) \wedge \|\mathbf{a}\|_{\infty} \leq 1 \right\} = \frac{1}{c^n} \|\mathbf{d}\|_1$$

where the last equality follows from $c_0(\mathbb{Z}^n, K)^* =_{\text{n.s.}} l^1(\mathbb{Z}^n, K)$. \square

Lemma 3.13. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $E =_{\text{n.s.}} C_u(\mathbb{R}^n, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. Let $c \in \mathbb{R}_+$, $\tilde{f} \in E^*$, $f \in C_{\text{com}}(\mathbb{R}^n, K)$, and $\mathbf{d}[\mathbf{k}] = \langle \tilde{f}, f(c \cdot -\mathbf{k}) \rangle$ for all $\mathbf{k} \in \mathbb{Z}^n$. Then $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$ and $\|\mathbf{d}\|_1 \leq N_{\text{cover}}(f) \|f\|_{\infty} \|\tilde{f}\|$.*

Proof. Suppose that $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. Let

$$g := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k}).$$

By Lemma 3.7 we have $g \in C_0(\mathbb{R}^n, K)$ and $\|g\|_{\infty} \leq N_{\text{cover}}(f) \|f\|_{\infty} \|\mathbf{a}\|_{\infty}$. Now

$$\langle \tilde{f}, g \rangle = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \langle \tilde{f}, f(c \cdot -\mathbf{k}) \rangle = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \mathbf{d}[\mathbf{k}] \in K$$

where the series converge absolutely. By Lemma 3.7 we have

$$\left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \mathbf{d}[\mathbf{k}] \right| = |\langle \tilde{f}, g \rangle| \leq \|\tilde{f}\| \|g\| \leq N_{\text{cover}}(f) \|f\|_{\infty} \|\tilde{f}\| \|\mathbf{a}\|_{\infty}. \quad (11)$$

As $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$ was arbitrary it follows that $\mathbf{d} \in c_0(\mathbb{Z}^n, K)^* =_{\text{n.s.}} l^1(\mathbb{Z}^n, K)$. It follows from Equation (11) that $\|\mathbf{d}\|_1 \leq N_{\text{cover}}(f) \|f\|_{\infty} \|\tilde{f}\|$. \square

Lemma 3.14. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $c \in \mathbb{R}_+$ and $f \in C_0(\mathbb{R}^n, K)$. Then $(f(c\mathbf{k}))_{\mathbf{k} \in \mathbb{Z}^n} \in c_0(\mathbb{Z}^n, K)$ and $\|(f(c\mathbf{k}))_{\mathbf{k} \in \mathbb{Z}^n}\|_{\infty} \leq \|f\|_{\infty}$.*

Proof. Let $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection. Let $h > 0$. There exists $r > 0$ so that $|f(\mathbf{x})| < h$ for all $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\| > r$. The set $\{\mathbf{k} \in \mathbb{Z}^n : \|\mathbf{c}\mathbf{k}\|_2 \leq r\}$ is finite and hence there exists $l_0 \in \mathbb{N}$ so that $\|c\sigma(l)\|_2 > r$ for all $l > l_0$. Now $|f(c\sigma(l))| < h$ for all $l > l_0$. Consequently $(f(c\mathbf{k}))_{\mathbf{k} \in \mathbb{Z}^n} \in c_0(\mathbb{Z}^n, K)$. It follows from the definition of the supremum norm that $\|(f(c\mathbf{k}))_{\mathbf{k} \in \mathbb{Z}^n}\|_{\infty} \leq \|f\|_{\infty}$. \square

Lemma 3.15. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $E =_{\text{n.s.}} C_u(\mathbb{R}^n, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. Let $c \in \mathbb{R}_+$, $\tilde{f} \in E^*$, and $f \in C_{\text{com}}(\mathbb{R}^n, K)$. Suppose that*

$$\forall \mathbf{k} \in \mathbb{Z}^n : \langle \tilde{f}, f(\cdot - \mathbf{k}) \rangle = \delta_{\mathbf{k}, 0}. \quad (12)$$

Then

$$\frac{1}{c^n} \frac{1}{N_{\text{cover}}(f) \|f\|_{\infty}} \|\mathbf{d}\|_1 \leq \left\| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k}) \right\|_{E^*} \leq \frac{1}{c^n} \|\tilde{f}\| \|\mathbf{d}\|_1$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$.

Proof. Let $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$. Let

$$\tilde{g} := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k}).$$

By Lemma 3.11 the series on the right-hand side converges absolutely and

$$\|\tilde{g}|E^*\| \leq \frac{1}{c^n} \|\tilde{f}|E^*\| \|\mathbf{d}\|_1.$$

Suppose that $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$. Let

$$g := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c \cdot -\mathbf{k})$$

By Lemma 3.7 the series on the right-hand side converges unconditionally and $\|g\|_\infty \leq N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty$. By Definition 2.16 we have

$$\langle \tilde{f}(c \cdot -\mathbf{k}), f(c \cdot -\ell) \rangle = \frac{1}{c^n} \langle \tilde{f}, f(\cdot + \mathbf{k} - \ell) \rangle$$

and hence by Equation (12)

$$\langle \tilde{g}, g \rangle = \frac{1}{c^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \sum_{\ell \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}[\ell] \delta_{\mathbf{k}, \ell} = \frac{1}{c^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}[\mathbf{k}]$$

and it follows that

$$\frac{1}{c^n} \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}[\mathbf{k}] \right| \leq \|\tilde{g}\| \|g\|_\infty \leq \|\tilde{g}\| N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty.$$

Thus

$$\|\tilde{g}\| \geq \frac{1}{c^n} \frac{1}{N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}\|_\infty} \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}[\mathbf{k}] \right|. \quad (13)$$

Let $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection. When $m \in \mathbb{N}$ define sequence $\mathbf{a}_m \in c_0(\mathbb{Z}^n, K)$ by

$$\mathbf{a}_m[\mathbf{k}] := \begin{cases} e^{-i \arg \mathbf{d}[\mathbf{k}]} & \text{if } \sigma^{-1}(\mathbf{k}) \leq m \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathbf{k} \in \mathbb{Z}^n$. Sequence $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$ was arbitrary and it follows from (13) that

$$\begin{aligned} \|\tilde{g}\| &\geq \frac{1}{c^n} \frac{1}{N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}_m\|_\infty} \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \mathbf{a}_m[\mathbf{k}] \right| \\ &= \frac{1}{c^n} \frac{1}{N_{\text{cover}}(f) \|f\|_\infty \|\mathbf{a}_m\|_\infty} \sum_{u=0}^m |\mathbf{d}[\sigma(u)]| \end{aligned}$$

for all $m \in \mathbb{N}$. Now $\|\mathbf{a}_m\|_\infty \leq 1$ for all $m \in \mathbb{N}$ and hence

$$\|\tilde{g}|E^*\| \geq \frac{1}{c^n} \frac{1}{N_{\text{cover}}(f) \|f\|_\infty} \sum_{u=0}^m |\mathbf{d}[\sigma(u)]|$$

for all $m \in \mathbb{N}$. Consequently

$$\|\tilde{g}|E^*\| \geq \frac{1}{c^n} \frac{1}{N_{\text{cover}}(f) \|f\|_\infty} \sum_{\mathbf{k} \in \mathbb{Z}^n} |\mathbf{d}[\mathbf{k}]| = \frac{1}{c^n} \frac{1}{N_{\text{cover}}(f) \|f\|_\infty} \|\mathbf{d}\|_1.$$

□

Lemma 3.16. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $E =_{\text{n.s.}} C_u(\mathbb{R}^n, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. Let $c \in \mathbb{R}_+$, $\tilde{f} \in E^*$, and $f \in C_{\text{com}}(\mathbb{R}^n, K)$. Suppose that

$$\langle \tilde{f}, f(\cdot - \mathbf{k}) \rangle = \delta_{\mathbf{k}, 0}$$

for all $\mathbf{k} \in \mathbb{Z}^n$. Define

$$M := \left\{ c^n \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k}) : \mathbf{d} \in l^1(\mathbb{Z}^n, K) \right\}$$

$$\|\tilde{g}|M\| := \|\tilde{g}|E^*\|. \quad (14)$$

Define function $\iota : l^1(\mathbb{Z}^n, K) \rightarrow M$ by

$$\iota(\mathbf{d}) := c^n \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{f}(c \cdot -\mathbf{k})$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$. Then

- (i) M is a closed subspace of E^* .
- (ii) Function ι is a topological isomorphism from $l^1(\mathbb{Z}^n, K)$ onto M and

$$\frac{1}{N_{\text{cover}}(f)\|f\|_\infty} \|\mathbf{d}\|_1 \leq \|\iota(\mathbf{d})\| \leq \|\tilde{f}\| \|\mathbf{d}\|_1 \quad (15)$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$.

Proof. The series in Equation (14) converges absolutely by Lemma 3.11. Function ι is linear and a surjection onto M . By Lemma 3.15 Equation (15) is true.

Suppose that $\mathbf{w}, \mathbf{z} \in l^1(\mathbb{Z}^n, K)$ and $\mathbf{w} \neq \mathbf{z}$. Now $\|\mathbf{w} - \mathbf{z}\|_1 \neq 0$ so

$$\|\iota(\mathbf{w} - \mathbf{z})\| \geq \frac{1}{N_{\text{cover}}(f)\|f\|_\infty} \|\mathbf{w} - \mathbf{z}\|_1 > 0.$$

It follows that $\iota(\mathbf{w} - \mathbf{z}) \neq 0$. Hence ι is an injection. It follows that ι is a linear bijection and vector spaces M and $l^1(\mathbb{Z}^n, K)$ are algebraically isomorphic. By Equation (15) ι and ι^{-1} are continuous. Thus (ii) is true.

Since M and $l^1(\mathbb{Z}^n, K)$ are topologically isomorphic and $l^1(\mathbb{Z}^n, K)$ is a Banach space it follows that M is also a Banach space and consequently a closed subspace of E^* . Thus (i) is true. \square

Lemma 3.17. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $E =_{\text{n.s.}} C_u(\mathbb{R}^n, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. Let $c \in \mathbb{R}_+$. Define

$$M := \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \delta \left(\cdot - \frac{\mathbf{k}}{c} \right) : \mathbf{d} \in l^1(\mathbb{Z}^n, K) \right\}$$

$$\|\tilde{g}|M\| := \|\tilde{g}|E^*\| \quad (16)$$

Define function $\iota : l^1(\mathbb{Z}^n, K) \rightarrow M$ by

$$\iota(\mathbf{d}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \delta \left(\cdot - \frac{\mathbf{k}}{c} \right)$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$. Then M is a closed subspace of E^* and function ι is an isometric isomorphism from $l^1(\mathbb{Z}^n, K)$ onto M .

Proof. See also [19, theorem 2.6]. The series in Equation (16) converges absolutely by Lemma 3.11. Function ι is linear and a surjection onto M . By Lemma 3.12 $\|\iota(\mathbf{d})\| = \|\mathbf{d}\|_1$ for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$. It follows that ι is injective. Hence ι is an isometric isomorphism from $l^1(\mathbb{Z}^n, K)$ onto M and M is a closed subspace of E^* . \square

Lemma 3.18. *Let $m \in \mathbb{N}$. Let $\mathbf{a} \in \mathbb{R}_0^{Z_0(m) \times \mathbb{N}}$ and $\mathbf{b}_k := (\mathbf{a}[k, l])_{l=0}^\infty$ for each $k \in Z_0(m)$. Then*

(i) *Series*

$$\sum_{\lambda \in Z_0(m) \times \mathbb{N}} \mathbf{a}[\lambda]$$

converges if and only if the series

$$\sum_{l \in \mathbb{N}} \mathbf{b}_k[l]$$

converges for all $k \in Z_0(m)$.

(ii)

$$\sum_{\lambda \in Z_0(m) \times \mathbb{N}} \mathbf{a}[\lambda] = \sum_{k \in Z_0(m)} \sum_{l \in \mathbb{N}} \mathbf{b}_k[l]$$

(iii) *When $p \in [1, \infty]$*

$$\|\mathbf{a}\|_p = \left\| \left(\|\mathbf{b}_k\|_p \right)_{k \in Z_0(m)} \right\|_p.$$

Lemma 3.19. *Let $n \in \mathbb{Z}_+$, $\mathbf{a} \in l^1$, $v_k \in C_{\text{com}}(\mathbb{R}^n)$ for each $k \in \mathbb{N}$, and*

$$f(\mathbf{x}) := \sum_{k \in \mathbb{N}} \mathbf{a}[k] v_k(\mathbf{x}) \tag{17}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Suppose also that

(i) *The series in (17) converges absolutely for each $\mathbf{x} \in \mathbb{R}^n$.*

(ii) *There exists $c_1 \in \mathbb{R}_+$ so that $\|v_k\|_1 \leq c_1$ for all $k \in \mathbb{N}$.*

Then

$$\int_{\mathbf{x} \in \mathbb{R}^n} \left(\sum_{k \in \mathbb{N}} |\mathbf{a}[k]| |v_k(\mathbf{x})| \right) d\tau = \sum_{k \in \mathbb{N}} |\mathbf{a}[k]| \int_{\mathbf{x} \in \mathbb{R}^n} |v_k(\mathbf{x})| d\tau.$$

Lemma 3.20. *Let $n \in \mathbb{Z}_+$, $f \in C_{\text{com}}(\mathbb{R}^n)$, $c \in \mathbb{R}_+$, and $\mathbf{a} \in l^\infty(\mathbb{Z}^n)$. Then the series*

$$g(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(c\mathbf{x} - \mathbf{k})$$

converges absolutely for each $\mathbf{x} \in \mathbb{R}^n$. We also have $g \in C_u(\mathbb{R}^n)$.

Lemma 3.21. *Let $n \in \mathbb{Z}_+$ and I be a countably infinite set. Let $v_\alpha \in C_{\text{com}}(\mathbb{R}^n)$ and $\tilde{v}_\alpha \in C_{\text{com}}(\mathbb{R}^n)$ for all $\alpha \in I$. When $\alpha \in I$ define $J(\alpha) := \{\beta \in I : \text{supp } \tilde{v}_\alpha \cap \text{supp } v_\beta \neq \emptyset\}$. Assume that $J(\alpha)$ is finite for each $\alpha \in I$. Let $\mathbf{a} \in l^\infty(I)$ and define*

$$f(\mathbf{x}) := \sum_{\alpha \in I} \mathbf{a}[\alpha] v_\alpha(\mathbf{x}) \tag{18}$$

for all $\mathbf{x} \in \mathbb{R}^n$. Assume that the series in Equation (18) converges absolutely for each $\mathbf{x} \in \mathbb{R}^n$. Then

$$\langle \tilde{v}_\alpha, f \rangle = \sum_{\beta \in I} \mathbf{a}[\beta] \langle \tilde{v}_\alpha, v_\beta \rangle.$$

Lemma 3.22. Let $n \in \mathbb{Z}_+$, $p \in [1, \infty[$, $x_k \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $y \in L^p(\mathbb{R}^n)$, and $z \in L^\infty(\mathbb{R}^n)$. Suppose that $\|x_k - y\|_{L^p(\mathbb{R}^n)} \rightarrow 0$ as $k \rightarrow \infty$ and $\|x_k - z\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0$ as $k \rightarrow \infty$. Then $y = z$ almost everywhere.

Lemma 3.23. Let (R, μ) and (S, ν) be totally σ -finite measure spaces. Suppose that $A_1 \subset_{c.s.} L^1(R, \mu)$, $A_\infty \subset_{c.s.} L^\infty(R, \mu)$, T_1 is an operator from A_1 into $L^1(S, \nu)$, and T_∞ is an operator from A_∞ into $L^\infty(S, \nu)$. When $p \in]1, \infty[$ define $T_p := (T_1, T_\infty)_{1-\frac{1}{p}, p}$, $A_p := \text{set } (A_1, A_\infty)_{1-\frac{1}{p}, p}$, and $\|f\|_{A_p} := \|f\|_{L^p(R, \mu)}$ for all $f \in A_p$. Then

$$\|T_p|_{\mathcal{L}(A_p, L^p(S, \nu))}\| \leq \frac{p}{p-1} \|T_1\|^{\frac{1}{p}} \|T_\infty\|^{1-\frac{1}{p}}$$

for all $p \in]1, \infty[$.

Proof. Use definitions IV.4.4 and V.1.7, lemma IV.4.5, and theorems V.1.6 and V.1.12 in [3]. \square

Definition 3.24. Let $f \in C_{\text{com}}(\mathbb{R}^n)$, $r_1 \in \mathbb{R}_+$, and $\tilde{f} \in C(\overline{B_{\mathbb{R}^n}}(0; r_1))^*$. Let $m \in \mathbb{N}$. We say that pair (\tilde{f}, f) spans all the polynomials of degree at most m iff

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \left\langle \tilde{f}(\cdot - \mathbf{k}), p|_{\overline{B_{\mathbb{R}^n}}(\mathbf{k}; r_1)} \right\rangle f(\mathbf{x} - \mathbf{k}) = p(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and for all the polynomials p of n variables that are of degree at most m .

Definition 3.25. Define function $\sigma_{\text{pl}} : \mathbb{N} \rightarrow \mathbb{Z}^2$ by setting $\sigma_{\text{pl}}(0) = (0, 0)$ and

$$\sigma_{\text{pl}}(k) := \begin{cases} (-m + \alpha, -m - 1); & k = (2m + 1)^2 + l, \ m \in \mathbb{N}, \ \alpha \in Z_0(2m + 1) \\ (m + 1, -m - 1 + \alpha); & k = (2m + 1)^2 + 2m + 1 + \alpha, \ m \in \mathbb{N}, \ \alpha \in Z(2m + 2) \\ (m + 1 - \alpha, m + 1); & k = (2m + 1)^2 + 4m + 3 + \alpha, \ m \in \mathbb{N}, \ \alpha \in Z(2m + 2) \\ (-m - 1, m + 1 - \alpha); & k = (2m + 1)^2 + 6m + 5 + \alpha, \ m \in \mathbb{N}, \ \alpha \in Z(2m + 2) \end{cases}$$

for all $k \in \mathbb{Z}_+$.

Function σ_{pl} is a bijection from \mathbb{N} onto \mathbb{Z}^2 . We give this fact without a proof.

Definition 3.26. Let $n \in \mathbb{N} + 2$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$. We say that \mathbf{a} and \mathbf{b} are *neighbours* iff $\|\mathbf{a} - \mathbf{b}\|_\infty = 1$.

Definition 3.27. Let $n \in \mathbb{Z}_+$, $I \subset \mathbb{N}$, $I \neq \emptyset$. Let $\alpha \in (\mathbb{Z}^n)^I$. We say that α *preserves neighbours* iff whenever $k \in \mathbb{N}$, $k, k + 1 \in I$ we have $\|\alpha(k + 1) - \alpha(k)\|_\infty = 1$.

Definition 3.28. When $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$ define $A_{n,k} := (Z_\pm(k))^n$ and $B_{n,k} := A_{n,k+1} \setminus A_{n,k}$.

Definition 3.29. Let $n \in \mathbb{Z}_+$ and $\eta : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection. We say that η is *increasing along zero-centred cubes* if $\eta[\#A_{n,k} + Z_0(\#B_{n,k} - 1)] = B_{n,k}$ for all $k \in \mathbb{N}$ and $\eta(0) = \mathbf{0}_n$.

We will prove that for each $n \in \mathbb{N} + 2$ there exists functions $\beta_{n,k} : Z_0(\#B_{n,k} - 1) \rightarrow B_{n,k}$, $k \in \mathbb{N}$, satisfying conditions

- (I1) Function $\beta_{n,k}$ is a bijection from $Z_0(\#B_{n,k} - 1)$ onto $B_{n,k}$.
- (I2) Function $\beta_{n,k}$ preserves neighbours.
- (I3) $\beta_{n,k}(\#B_{n,k} - 1) = (-k - 1)\mathbf{1}_n$
- (I4) Points $\beta_{n,k}(0)$ and $\beta_{n,k}(\#B_{n,k} - 1)$ are neighbours.
- (I5) Points $\beta_{n,k}(0)$ and $\beta_{n,k}(\#B_{n,k} - 2)$ are neighbours.
- (I6) Points $\beta_{n,k}(0)$ and $\beta_{n,k+1}(0)$ are neighbours.

(I7)

(I7.1) If $k \geq 1$ points $\beta_{n,k}(0)$ and $\beta_{n,k-1}(\#B_{n,k-1} - 1)$ are neighbours.

(I7.2) If $k = 0$ points $\beta_{n,0}(0)$ and $\mathbf{0}_n$ are neighbours.

(I8)

(I8.1) If $k \geq 1$ points $\beta_{n,k}(1)$ and $\beta_{n,k-1}(\#B_{n,k-1} - 1)$ are neighbours.

(I8.2) If $k = 0$ points $\beta_{n,0}(1)$ and $\mathbf{0}_n$ are neighbours.

Lemma 3.30. *Let $k \in \mathbb{N}$. Define $\beta_{2,k} := \sigma_{\text{pl}}(\cdot + (2k+1)^2)|_{Z_0(8k+7)}$. Then functions $\beta_{2,k}$ satisfy conditions (I1) – (I8).*

Lemma 3.31. *Let $n \in \mathbb{N} + 2$. There exist functions $\beta_{n,k} : Z_0(\#B_{n,k} - 1) \rightarrow B_{n,k}$, for each $k \in \mathbb{N}$, satisfying conditions (I1) – (I8).*

Proof. Case $n = 2$ has been proved by Lemma 3.30. Assume that the proposition is true for some $n \in \mathbb{N} + 2$ and for all $k \in \mathbb{N}$. We will then prove the case $n + 1$ for some arbitrary $k \in \mathbb{N}$. Define

$$\begin{aligned}
V' &:= \{\beta_{n,l}(0) : l \in Z_0(k)\} \\
V &:= \{s_{\text{comb}}(-k-1, \mathbf{v}) : \mathbf{v} \in V'\} \\
Q &:= \{(-k-1)\mathbf{1}_{n+1}\} \\
R &:= \{\mathbf{v} \in (Z_{\pm}(k+1))^{n+1} : \mathbf{v}[1] = -k-1\} \\
R' &:= R \setminus (V \cup Q) \\
S(t) &:= \{s_{\text{comb}}(t, \mathbf{v}) : \mathbf{v} \in (Z_{\pm}(k+1))^n \wedge \exists j_1 \in Z(n) : |\mathbf{v}[j_1]| = k+1\}, \quad t \in Z_{\pm}(k) \\
S'(t) &:= S(t) \setminus \{s_{\text{comb}}(t, -(k+1)\mathbf{1}_n)\}, \quad t \in Z_{\pm}(k) \\
Y &:= \{s_{\text{comb}}(k+1, \mathbf{v}) : \mathbf{v} \in V'\} \\
T &:= \{\mathbf{v} \in (Z_{\pm}(k+1))^{n+1} : \mathbf{v}[1] = k+1\} \\
T' &:= T \setminus Y \\
W &:= \{s_{\text{comb}}(b, (-k-1)\mathbf{1}_n) : b \in Z_{\pm}(k)\}
\end{aligned}$$

for all $k \in \mathbb{N}$.

Order the elements in the following order: $V, R', S'(-k), \dots, S'(k), Y, T', W, Q$.

1. Order V by $s_{\text{comb}}(-k-1, \beta_{n,k-l}(0))$, $l \in 0, \dots, k$. By induction assumption (I6) each point in this ordering and its successor are neighbours. By induction assumptions (I3) and (I4) points $\beta_{n,k}(0)$ and $(-k-1)\mathbf{1}_n$ are neighbours, from which it follows that $\beta_{n,k}(0) \in \{-k-1, -k\}^n$. Hence $s_{\text{comb}}(-k-1, \beta_{n,k}(0))$ and $-k\mathbf{1}_{n+1}$ are neighbours.

2. Order R' by ordering the last n coordinates of the points with function $\alpha_{n,k+1}$ starting from $s_{\text{comb}}(-k-1, \mathbf{0}_n)$ and ending to $(-k-1)\mathbf{1}_{n+1}$ and skipping the last element and the elements belonging to V .

Let $\mathbf{x}_0 := s_{\text{comb}}(-k-1, \beta_{n,k-l_0}(0))$ for some $l_0 \in Z_0(k)$. Let \mathbf{x}_1 be the successor of point \mathbf{x}_0 in the set R , $\mathbf{x}_1 = s_{\text{comb}}(-k-1, \beta_{n,k-l_0}(1)) \notin V$. Let \mathbf{x}_{-1} be the predecessor of the point \mathbf{x}_0 in the set R . If $l_0 < k$ we have $\mathbf{x}_{-1} = s_{\text{comb}}(-k-1, \beta_{n,k-l_0-1}(\#B_{n,k-l_0-1} - 1)) \notin V$ and by the induction assumption (I8.1) points \mathbf{x}_{-1} and \mathbf{x}_1 are neighbours. If $l_0 = k$ we have $\mathbf{x}_{-1} = s_{\text{comb}}(-k-1, \mathbf{0}_n) \notin V$ and by the induction assumption (I8.2) points \mathbf{x}_{-1} and \mathbf{x}_1 are neighbours. Denote the last element in the ordering by $\mathbf{x}_{\text{last}} := s_{\text{comb}}(-k-1, \beta_{n,k}(\#B_{n,k} - 2))$.

3. Order $S'(-k)$ by ordering the last n coordinates of the points with the function $\beta_{n,k}$ and skipping point $s_{\text{comb}}(-k, -(k+1)\mathbf{1}_n)$. Now the first element is $\mathbf{y}_0 := s_{\text{comb}}(-k, \beta_{n,k}(0))$ and the last element $s_{\text{comb}}(-k, \beta_{n,k}(\#B_{n,k} - 2))$. By the induction assumption (I5) points $\beta_{n,k}(0)$ and $\beta_{n,k}(\#B_{n,k} - 2)$ are neighbours, from which it follows that points \mathbf{y}_0 and \mathbf{x}_{last} are neighbours.
4. If $k = 0$ this step is left away. Order $S'(t)$, $t = -k+1, \dots, k$ by ordering the last n coordinates of the points with the function $\beta_{n,k}$ and skipping point $s_{\text{comb}}(t, -(k+1)\mathbf{1}_n)$. Now the first element is $\mathbf{z}_0 := s_{\text{comb}}(t, \beta_{n,k}(0))$ and the last element $s_{\text{comb}}(t, \beta_{n,k}(\#B_{n,k} - 2))$. The predecessor of the point \mathbf{z}_0 in the set $S'(t-1)$ is $\mathbf{z}_{-1} := s_{\text{comb}}(t-1, \beta_{n,k}(\#B_{n,k} - 2))$.
By the induction assumption (I5) points $\beta_{n,k}(0)$ and $\beta_{n,k}(\#B_{n,k} - 2)$ are neighbours, from which it follows that points \mathbf{z}_0 and \mathbf{z}_{-1} are neighbours.
5. Order Y by $s_{\text{comb}}(-k-1, \beta_{n,k-l}(0))$, $l \in 0, \dots, k$. By induction assumption (I6) each point in this ordering and its successor are neighbours.
6. Order T' by ordering the last n coordinates of the points with function $\alpha_{n,k+1}$ starting from $s_{\text{comb}}(k+1, \mathbf{0}_n)$ and ending to $s_{\text{comb}}(k+1, (-k-1)\mathbf{1}_n)$ and skipping the elements belonging to Y .
Let $\mathbf{w}'_0 := s_{\text{comb}}(-k-1, \beta_{n,k-l_0}(0))$ for some $l_0 \in Z_0(k)$. Let \mathbf{w}_1 be the successor of point \mathbf{w}'_0 in the set T , $\mathbf{w}_1 = s_{\text{comb}}(k+1, \beta_{n,k-l_0}(1)) \notin Y$. Let \mathbf{w}_{-1} be the predecessor of the point \mathbf{w}'_0 in the set T . If $l_0 < k$ we have $\mathbf{w}_{-1} = s_{\text{comb}}(k+1, \beta_{n,k-l_0-1}(\#B_{n,k-l_0-1} - 1)) \notin Y$ and by the induction assumption (I8.1) points \mathbf{w}_{-1} and \mathbf{w}_1 are neighbours. If $l_0 = k$ we have $\mathbf{w}_{-1} = s_{\text{comb}}(k+1, \mathbf{0}_n) \notin Y$ and by the induction assumption (I8.2) points \mathbf{w}_{-1} and \mathbf{w}_1 are neighbours. Denote the last element in the ordering by $\mathbf{w}_{\text{last}} := s_{\text{comb}}(k+1, (-k-1)\mathbf{1}_n)$.
7. Order W with $s_{\text{comb}}(k-l, (-k-1)\mathbf{1}_n)$ for $l \in 0, \dots, 2k$. The first element $s_{\text{comb}}(k, (-k-1)\mathbf{1}_n)$ and \mathbf{w}_{last} are neighbours.
8. Set $(-k-1)\mathbf{1}_{n+1}$ as the last element. The predecessor of this point in the set W is $s_{\text{comb}}(-k, (-k-1)\mathbf{1}_n)$

We have constructed ordering $\beta_{n+1,k}$ for the set $B_{n+1,k}$ so that $\beta_{n+1,k}$ satisfies conditions (I1) and (I2). Condition (I3) is satisfied by the construction of $\beta_{n+1,k}$, step 8.

By the construction of $\beta_{n+1,k}$, step 1, we have $\beta_{n+1,k}(0) = s_{\text{comb}}(-k-1, \beta_{n,k}(0))$. By the construction, step 8, and induction assumption (I3) we have $\beta_{n+1,k}(\#B_{n+1,k} - 1) = (-k-1)\mathbf{1}_{n+1} = s_{\text{comb}}(-k-1, \beta_{n,k}(\#B_{n,k} - 1))$. By the induction assumption (I4) points $\beta_{n,k}(0)$ and $\beta_{n,k}(\#B_{n,k} - 1)$ are neighbours. It follows that points $\beta_{n+1,k}(0)$ and $\beta_{n+1,k}(\#B_{n+1,k} - 1)$ are neighbours. Hence (I4) is true for $n+1$.

By the construction, steps 7 and 8, we have $\beta_{n+1,k}(\#B_{n+1,k} - 1) = (-k-1)\mathbf{1}_{n+1}$ and

$$\beta_{n+1,k}(\#B_{n+1,k} - 2) = s_{\text{comb}}(-k, (-k-1)\mathbf{1}_n) \quad (19)$$

By (I4) in case $n+1$, which has been proved, points $\beta_{n+1,k}(0)$ and $\beta_{n+1,k}(\#B_{n+1,k} - 1)$ are neighbours. By condition (I3) in case $n+1$, which has been proved, we have $\beta_{n+1,k}(\#B_{n+1,k} - 1) = (-k-1)\mathbf{1}_{n+1}$. Consequently, points $\beta_{n+1,k}(0)$ and $(-k-1)\mathbf{1}_{n+1}$ are neighbours. As $\beta_{n+1,k}(0) \in B_{n,k}$ it follows that

$$\beta_{n+1,k}(0) \in \{-k-1, -k\}^{n+1}. \quad (20)$$

By Equations (19) and (20) points $\beta_{n+1,k}(0)$ and $\beta_{n+1,k}(\#B_{n+1,k} - 2)$ are neighbours. Thus condition (I5) is true for $n+1$.

We will next prove (I6) by induction. When $n = 2$ we have $\beta_{2,k}(0) = (-k, -k-1)$ for all $k \in \mathbb{N}$. It follows that points $\beta_{2,k}(0)$ and $\beta_{2,k+1}(0) = (-k-1, -k-2)$ are neighbours. Suppose then that (I6) is true for some $n \in \mathbb{N} + 2$ and for all $k \in \mathbb{N}$. By construction, step 1, we have

$$\beta_{n+1,k}(0) = s_{\text{comb}}(-k-1, \beta_{n,k}(0)) \quad (21)$$

and

$$\beta_{n+1,k+1}(0) = s_{\text{comb}}(-k-2, \beta_{n,k+1}(0)). \quad (22)$$

By the induction assumption points $\beta_{n,k}(0)$ and $\beta_{n,k+1}(0)$ are neighbours. Hence by Equations (21) and (22) points $\beta_{n+1,k}(0)$ and $\beta_{n+1,k+1}(0)$ are neighbours. Thus we have shown (I6) in case $n+1$.

We will next show (I7). Assume first that $k \geq 1$ and show (I7.1). By (I3) in case $n+1$, which has already been proved, we have

$$\beta_{n+1,k-1}(\#B_{n,k-1} - 1) = -k\mathbf{1}_{n+1}. \quad (23)$$

By (I4) in case $n+1$, which has already been proved, points $\beta_{n+1,k}(0)$ and $\beta_{n+1,k}(\#B_{n+1,k} - 1)$ are neighbours. Thus $\beta_{n+1,k}(0)$ and $(-k-1)\mathbf{1}_{n+1}$ are neighbours. As $\beta_{n+1,k}(0) \in B_{n+1,k}$ it follows that

$$\beta_{n+1,k}(0) \in \{-k-1, -k\}^{n+1}. \quad (24)$$

It follows from Equations (23) and (24) that the points $\beta_{n+1,k}(0)$ and

$$\beta_{n+1,k-1}(\#B_{n+1,k-1} - 1)$$

are neighbours. Hence (I7.1) is true. Assume then that $k = 0$ and show (I7.2). By condition (I4) in case $n+1$, which has been proved, points $\beta_{n+1,0}(0)$ and $\beta_{n+1,0}(\#B_{n+1,0} - 1)$ are neighbours. By condition (I3) in case $n+1$, which has been proved, $\beta_{n+1,0}(\#B_{n+1,0} - 1) = -\mathbf{1}_{n+1}$. As $\beta_{n+1,0}(0) \in B_{n+1,0}$ it follows that $\beta_{n+1,0}(0) \in \{-1, 0\}^{n+1}$. Consequently $\beta_{n+1,0}(0)$ and $\mathbf{0}_{n+1}$ are neighbours. Thus (I7.2) is true.

We will next show (I8). Assume first that $k \geq 1$ and show (I8.1). By the construction, step 1, points $\beta_{n+1,k}(0)$ and $(-k-1)\mathbf{1}_{n+1}$ are neighbours. It follows that $\beta_{n+1,k}(0) \in \{-k-1, -k\}^{n+1}$ and

$$\beta_{n+1,k}(1) \in \{-k-1, -k, -k+1\}^{n+1} \quad (25)$$

By (I3) we have $\beta_{n+1,k-1}(\#B_{n+1,k-1} - 1) = -k\mathbf{1}_{n+1}$. By Equation (25) points $\beta_{n+1,k}(1)$ and

$$\beta_{n+1,k-1}(\#B_{n+1,k-1} - 1)$$

are neighbours. Thus (I8.1) is true. Assume then that $k = 0$ and show (I8.2). By construction, steps 1 and 2, we have $\beta_{n+1,0}(1) = s_{\text{comb}}(-1, \mathbf{0}_n)$. It follows that $\beta_{n+1,0}(1)$ and $\mathbf{0}_{n+1}$ are neighbours. Thus (I8.2) is true.

Function $\beta_{n+1,k} : Z_0(\#B_{n+1,k} - 1) \rightarrow B_{n+1,k}$ is a bijection and preserves neighbours. Construct an ordering $\alpha_{n+1,k+1}$ for the set $A_{n+1,k+1}$ by putting $B_{n+1,k}$ ordered by $\beta_{n+1,k}$ after $A_{n+1,k}$ ordered by $\alpha_{n+1,k}$. Function $\alpha_{n+1,k+1}$ is a bijection, preserves neighbours, and increases along zero-centred cubes. \square

Define function $\sigma_c^{[n]} : \mathbb{N} \rightarrow \mathbb{Z}^n$ by setting

$$\sigma_c^{[n]}(m) := \begin{cases} \beta_{n,k}(l); & (2k+1)^n \leq m = l + (2k+1)^n < (2k+3)^n \\ \mathbf{0}_n; & m = 0 \end{cases}$$

for all $m \in \mathbb{N}$. By Lemma 3.31 function $\sigma_c^{[n]} : \mathbb{N} \rightarrow \mathbb{Z}^n$ is a bijection, preserves neighbours, and increases along zero-centred cubes.

4 On Tensor Product Spaces

Definition 4.1. When $n \in \mathbb{Z}_+$ define function $\sigma_{\text{sq}}^{[n]} : \mathbb{N} \rightarrow \mathbb{N}^n$ by

$$\sigma_{\text{sq}}^{[n]}(k) := \begin{cases} k; & n = 1 \\ s_{\text{comb}}(\sigma_{\text{sq}}^{[n-1]}(\sigma_{\text{sq}1}(k)), \sigma_{\text{sq}2}(k)); & n \geq 2 \end{cases}$$

for all $k \in \mathbb{N}$. When $n \in \mathbb{Z}_+$ and $j \in Z(n)$ define function $\sigma_{\text{sq}}^{(n,j)} : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\sigma_{\text{sq}}^{(n,j)}(k) := \sigma_{\text{sq}}^{[n]}(k)[j]$$

for all $k \in \mathbb{N}$.

It can be shown by induction that function $\sigma_{\text{sq}}^{[n]}$ is a bijection from \mathbb{N} onto \mathbb{N}^n for each $n \in \mathbb{Z}_+$.

Lemma 4.2. *Let $n \in \mathbb{Z}_+$ and $\alpha = \pi$ or $\alpha = \varepsilon$. Suppose that A_k is a Banach space for each $k \in Z(n)$. Suppose also that $(a_{k,j})_{j=0}^\infty$ is a Schauder basis of Banach space A_k for each $k \in Z(n)$. Then the sequence*

$$\left(\bigotimes_{k=1}^n a_{k, \sigma_{\text{sq}}^{(n,k)}(j)} \right)_{j=0}^\infty$$

is a Schauder basis of Banach space

$$\bigotimes_{k=1}^n \alpha A_k.$$

Proof. Use the two dimensional case and induction by n . □

We give the following definition to be used with inherited tensor products.

Definition 4.3. Let A and B be Banach spaces and α a reasonable crossnorm on $A \otimes B$. We say that α *preserves operator continuity* iff $S \otimes T$ is an operator from $A \otimes_\alpha B$ into $A \otimes_\alpha B$ whenever $S : A \rightarrow A$ and $T : B \rightarrow B$ are operators.

If α is a uniform crossnorm $\alpha_{A,B}$ preserves operator continuity for all Banach spaces A and B . The distributive law for completed Banach space tensor products and direct sums is proved next. Relationship between tensor products and direct sums has also been investigated by Greco and Ryan [20] and Ansemil and Floret [2]. Some lemmas are proved first.

Lemma 4.4. *Let E be a Banach space. Let B and C be closed subspaces of E so that $B \cap C = \{0\}$. Define operators $P_B : B \dot{+} C \rightarrow B$ and $P_C : B \dot{+} C \rightarrow C$ by*

$$P_B(b + c) := b, \quad x = b + c \in B \dot{+} C$$

and

$$P_C(b + c) := c, \quad x = b + c \in B \dot{+} C.$$

Then $B \dot{+} C$ is a closed subspace of E .

Proof. The direct sum $B \dot{+} C$ is a normed subspace of E . Let $M = \max\{\|P_B\|, \|P_C\|\}$. Let $(x_k)_{k=0}^\infty \subset B \dot{+} C$ be a Cauchy sequence and $x_k = b_k + c_k$ where $b_k \in B$ and $c_k \in C$ for all $k \in \mathbb{N}$. Let $h > 0$. There exists $N \in \mathbb{N}$ so that $\|x_{k'} - x_k\| < \frac{h}{M}$ for all $k, k' \in \mathbb{N}$ so that $k, k' \geq N$. Now $\|b_{k'} - b_k\| = \|P_B(x_{k'} - x_k)\| \leq \|P_B\| \|x_{k'} - x_k\| < h$ when $k, k' \geq N$. Similarly $\|c_{k'} - c_k\| < h$ when $k, k' \geq N$. Hence $(b_k)_{k=0}^\infty$ is a Cauchy sequence in B and $(c_l)_{l=0}^\infty$ is a Cauchy sequence in C . So $b_k \rightarrow b \in B$ as $k \rightarrow \infty$ and $c_l \rightarrow c \in C$ as $l \rightarrow \infty$. Furthermore, $x_k = b_k + c_k \rightarrow b + c \in B \dot{+} C$ as $k \rightarrow \infty$. Hence every Cauchy sequence in $B \dot{+} C$ converges in that space and $B \dot{+} C$ is a closed subspace of E . □

Lemma 4.5. *Let A and E be Banach spaces. Let B and C be closed subspaces of E so that $B \cap C = \{0\}$. Let α be a reasonable crossnorm on $A \otimes (B \dot{+} C)$ and β a reasonable crossnorm on $(B \dot{+} C) \otimes A$. Suppose that α and β preserve operator continuity. Define P_B and P_C as in Lemma 4.4 and suppose that P_B and P_C are operators. Then $A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B \cap A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C = \{0\}$ and $B \hat{\otimes}_{\beta; A \otimes (B \dot{+} C)} A \cap C \hat{\otimes}_{\beta; A \otimes (B \dot{+} C)} A = \{0\}$.*

Proof. Suppose that P_B is continuous.

Function id_A is an operator. Define function $P_1 : A \otimes_\alpha (B \dot{+} C) \rightarrow A \otimes_{\alpha; A \otimes (B \dot{+} C)} B$ by $P_1 = \text{id}_A \otimes P_B$. Function P_1 is an operator and (see [38, chapter 6.1]) it has a unique continuous linear extension $P'_1 : A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} (B \dot{+} C) \rightarrow A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B$. P_1 and P'_1 have the same norm.

Let $f \in A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B$. There exists a sequence $(f_k)_{k=0}^\infty$ in the noncompleted tensor product $A \otimes_{\alpha; A \otimes (B \dot{+} C)} B$ so that $\|f - f_k\|_\alpha \rightarrow 0$ as $k \rightarrow \infty$. Now $P'_1 f_k = P_1 f_k = f_k$ for all $k \in \mathbb{N}$ and hence $P'_1 f_k \rightarrow f$ as $k \rightarrow \infty$. On the other hand, P'_1 is continuous and $P'_1 f_k \rightarrow P'_1 f$ as $k \rightarrow \infty$. Hence $P'_1 f = f$. Therefore $P'_1 f = f$ for all $f \in A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B$.

Let $g \in A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C$. There exists a sequence $(g_k)_{k=0}^\infty$ in the noncompleted tensor product $A \otimes_{\alpha; A \otimes (B \dot{+} C)} C$ so that $\|g - g_k\|_\alpha \rightarrow 0$ as $k \rightarrow \infty$. Now

$$g_k = \sum_{i=1}^{m(k)} a_{k,i} \otimes c_{k,i}$$

where $a_{k,i} \in A$, $c_{k,i} \in C$, $m(k) \in \mathbb{N}$, and $k \in \mathbb{N}$. Since $P_B c = 0$ for all $c \in C$

$$P_B g_k = \sum_{i=1}^{m(k)} a_{k,i} \otimes (P_B c_{k,i}) = 0 \quad \forall k \in \mathbb{N}.$$

Hence $P'_1 g = 0$. If $g \in A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C$ and $g \neq 0$ then $P'_1 g = 0$ and $P'_1 g \neq g$ and hence $g \notin A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B$. So $A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B \cap A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C = \{0\}$. Proof of equation $B \hat{\otimes}_{\beta; A \otimes (B \dot{+} C)} A \cap C \hat{\otimes}_{\beta; A \otimes (B \dot{+} C)} A = \{0\}$ is similar. \square

Theorem 4.6. Let A and E be Banach spaces. Let B and C be closed subspaces of E so that $B \cap C = \{0\}$. Define P_B and P_C as in Lemma 4.4 and suppose that P_B and P_C are operators. Let α be a reasonable crossnorm on $A \otimes (B \dot{+} C)$ and β a reasonable crossnorm on $(B \dot{+} C) \otimes A$. Suppose that α and β preserve operator continuity. Now $A \hat{\otimes}_\alpha (B \dot{+} C) =_{\text{n.s.}} A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B \dot{+} A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C$ and $(B \dot{+} C) \hat{\otimes}_\beta A =_{\text{n.s.}} B \hat{\otimes}_{\beta; (B \dot{+} C) \otimes A} A \dot{+} C \hat{\otimes}_{\beta; (B \dot{+} C) \otimes A} A$.

Proof. By Lemma 4.5

$$A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B \cap A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C = \{0\}.$$

Let

$$M :=_{\text{n.s.}} A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B \dot{+} A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C.$$

Now M is a normed subspace of $A \hat{\otimes}_\alpha (B \dot{+} C)$. Define operators $P_1 : A \otimes_\alpha (B \dot{+} C) \rightarrow A \otimes_\alpha B$ and $P_2 : A \otimes_\alpha (B \dot{+} C) \rightarrow A \otimes_\alpha C$ by $P_1 = \text{id}_A \otimes P_B$ and $P_2 = \text{id}_A \otimes P_C$. Operator P_1 has a unique continuous linear extension $P'_1 : A \hat{\otimes}_\alpha (B \dot{+} C) \rightarrow A \hat{\otimes}_\alpha B$. We have $\|P_1\| = \|P'_1\|$. Similarly, operator P_2 has a unique continuous linear extension $P'_2 : A \hat{\otimes}_\alpha (B \dot{+} C) \rightarrow A \hat{\otimes}_\alpha C$ and we have $\|P_2\| = \|P'_2\|$. Let $x \in M$. Now $x = u + v$ where $u \in A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B$ and $v \in A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C$. There exists $(u_k)_{k=0}^\infty \subset A \otimes_{\alpha; A \otimes (B \dot{+} C)} B$ so that $u_k \rightarrow u$ as $k \rightarrow \infty$ and

$$u_k = \sum_{j=1}^{m_1(k)} w_{j,k} \otimes b_{j,k}$$

where $m_1(k) \in \mathbb{Z}_+$, $w_{j,k} \in A$, and $b_{j,k} \in B$. There exists $(v_k)_{k=0}^\infty \subset A \otimes_{\alpha; A \otimes (B \dot{+} C)} C$ so that $v_k \rightarrow v$ as $k \rightarrow \infty$ and

$$v_k = \sum_{j=1}^{m_2(k)} z_{j,k} \otimes c_{j,k}$$

where $m_2(k) \in \mathbb{Z}_+$, $z_{j,k} \in A$, and $c_{j,k} \in C$. Now $P_1 u_k = u_k$ and $P_1 v_k = 0$ for all $k \in \mathbb{N}$ and hence $P'_1 u = u$ and $P'_1 v = 0$. Similarly, $P_2 u_k = 0$ and $P_2 v_k = v_k$ for all $k \in \mathbb{N}$ and hence $P'_2 u = 0$ and $P'_2 v = v$. So $P'_1 x = u$ and $P'_2 x = v$. By Lemma 4.4 the normed vector space M is a closed subspace of $A \hat{\otimes}_\alpha (B \dot{+} C)$.

Let $t \in A \hat{\otimes}_\alpha (B \dot{+} C)$. Let $h > 0$. There exists $y \in A \otimes_\alpha (B \dot{+} C)$ so that $\|y - t\| < h$. Now

$$y = \sum_{j=1}^n a_j \otimes (b'_j + c'_j) = \sum_{j=1}^n a_j \otimes b'_j + \sum_{j=1}^n a_j \otimes c'_j$$

where $n \in \mathbb{N}$ and $a_j \in A$, $b'_j \in B$, $c'_j \in C$ for $j \in Z(n)$. Therefore $y \in A \otimes_{\alpha; A \otimes (B \dot{+} C)} B \dot{+} A \otimes_{\alpha; A \otimes (B \dot{+} C)} C \subset M$. Number $h > 0$ was arbitrary so $t \in \overline{M} = M$. Hence

$$A \hat{\otimes}_\alpha (B \dot{+} C) =_{\text{n.s.}} A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} B \dot{+} A \hat{\otimes}_{\alpha; A \otimes (B \dot{+} C)} C.$$

Proof of

$$(B \dot{+} C) \hat{\otimes}_\beta A =_{\text{n.s.}} B \hat{\otimes}_{\beta; (B \dot{+} C) \otimes A} A \dot{+} C \hat{\otimes}_{\beta; (B \dot{+} C) \otimes A} A$$

is similar. \square

Corollary 4.7. *Let α be an injective uniform crossnorm. Let A and E be Banach spaces. Let B and C be closed subspaces of E so that $B \cap C = \{0\}$. Define P_B and P_C as in Lemma 4.4 and suppose that P_B and P_C are operators. Then*

$$\begin{aligned} A \hat{\otimes}_\alpha B \cap A \hat{\otimes}_\alpha C &= \{0\}, \\ B \hat{\otimes}_\alpha A \cap C \hat{\otimes}_\alpha A &= \{0\}, \end{aligned}$$

and

$$\begin{aligned} A \hat{\otimes}_\alpha (B \dot{+} C) &=_{\text{n.s.}} A \hat{\otimes}_\alpha B \dot{+} A \hat{\otimes}_\alpha C, \\ (B \dot{+} C) \hat{\otimes}_\alpha A &=_{\text{n.s.}} B \hat{\otimes}_\alpha A \dot{+} C \hat{\otimes}_\alpha A. \end{aligned}$$

Lemma 4.8. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $n, m \in \mathbb{Z}_+$, X be a closed subspace of $C_b(\mathbb{R}^n, K)$ and Y be a closed subspace of $C_b(\mathbb{R}^m, K)$. Then*

$$\varepsilon(u) = \sup_{\mathbf{t} \in \mathbb{R}^{n+m}} \left| \sum_{i=1}^k x_i(\mathbf{t}[1], \dots, \mathbf{t}[n]) y_i(\mathbf{t}[n+1], \dots, \mathbf{t}[n+m]) \right|$$

for all $u \in X \otimes Y$ where

$$u = \sum_{i=1}^k x_i \otimes y_i,$$

$k \in \mathbb{Z}_+$, and $x_i \in X$, $y_i \in Y$ for $i \in Z(k)$. I.e. the injective tensor norm of u equals to the norm of u as an element of $C_b(\mathbb{R}^{n+m}, K)$.

Proof. Let $u \in X \otimes Y$,

$$u = \sum_{i=1}^k x_i \otimes y_i,$$

where $k \in \mathbb{Z}_+$ and $x_i \in X$, $y_i \in Y$ for $i \in Z(k)$. Now

$$\varepsilon(u) = \sup \left\{ \left| \sum_{i=1}^k \varphi(x_i) \psi(y_i) \right| : \varphi \in B_{X^*}, \psi \in B_{Y^*} \right\}$$

Set $A_1 = \{\delta_{\mathbf{t}} \in X^* : \mathbf{t} \in \mathbb{R}^n\}$ is a norming set of X , set $A_2 = \{\delta_{\mathbf{t}} \in Y^* : \mathbf{t} \in \mathbb{R}^m\}$ is a norming set of Y , and consequently [38, chapter 3.1]

$$\begin{aligned} \varepsilon(u) &= \sup \left\{ \left| \sum_{i=1}^k \varphi(x_i) \psi(y_i) \right| : \varphi \in A_1, \psi \in A_2 \right\} = \sup \left\{ \left| \sum_{i=1}^k \delta_{\mathbf{t}_1}(x_i) \delta_{\mathbf{t}_2}(y_i) \right| : \mathbf{t}_1 \in \mathbb{R}^n, \mathbf{t}_2 \in \mathbb{R}^m \right\} \\ &= \sup_{\mathbf{t} \in \mathbb{R}^{n+m}} \left| \sum_{i=1}^k x_i(\mathbf{t}[1], \dots, \mathbf{t}[n]) y_i(\mathbf{t}[n+1], \dots, \mathbf{t}[n+m]) \right|. \end{aligned}$$

□

Theorem 4.9. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $n, m \in \mathbb{Z}_+$, X be a closed subspace of $C_b(\mathbb{R}^n, K)$, and Y be a closed subspace of $C_b(\mathbb{R}^m, K)$. The tensor product $X \hat{\otimes}_\varepsilon Y$ is a closed subspace of $C_b(\mathbb{R}^{n+m}, K)$ and consequently*

$$\varepsilon(u) = \|u\|_\infty \quad \forall u \in X \hat{\otimes}_\varepsilon Y$$

where $\|u\|_\infty$ is the supremum norm of u as an element of $C_b(\mathbb{R}^{n+m}, K)$.

Proof. The algebraic tensor product $X \otimes Y$ is isomorphic to a subspace of $C_b(\mathbb{R}^{n+m}, K)$ as a vector space. By Lemma 4.8 the norm in $X \otimes_\varepsilon Y$ is equal to the supremum norm. Therefore the completion $X \hat{\otimes}_\varepsilon Y$ is the closure of $X \otimes_\varepsilon Y$ in $C_b(\mathbb{R}^{n+m}, K)$. Hence the norm of an element $u \in X \hat{\otimes}_\varepsilon Y$ equals the norm of u in $C_b(\mathbb{R}^{n+m}, K)$. □

Corollary 4.10. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$, $n \in \mathbb{N} + 2$, and X_1, \dots, X_n be closed subspaces of $C_b(\mathbb{R}, K)$. Then the tensor product $X_1 \hat{\otimes}_\varepsilon \dots \hat{\otimes}_\varepsilon X_n$ is a closed subspace of $C_b(\mathbb{R}^n, K)$.*

Lemma 4.11. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $n, m \in \mathbb{Z}_+$, $f \in C_0(\mathbb{R}^n, K)$ and $g \in C_0(\mathbb{R}^m, K)$. Then $f \otimes g \in C_0(\mathbb{R}^{n+m}, K)$.*

Proof. Define operator $A_1 : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ by

$$A_1 \mathbf{x} := (\mathbf{x}[1], \dots, \mathbf{x}[n]), \quad \mathbf{x} \in \mathbb{R}^{n+m}$$

and operator $A_2 : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ by

$$A_2 \mathbf{x} := (\mathbf{x}[n+1], \dots, \mathbf{x}[n+m]), \quad \mathbf{x} \in \mathbb{R}^{n+m}.$$

Now $(f \otimes g)(\mathbf{x}) = f(A_1 \mathbf{x})g(A_2 \mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{n+m}$ and hence $f \otimes g$ is continuous. Since $|(f \otimes g)(\mathbf{x})| = |f(A_1 \mathbf{x})g(A_2 \mathbf{x})| \leq \|f\|_\infty \|g\|_\infty$ for all $\mathbf{x} \in \mathbb{R}^{n+m}$ function $f \otimes g$ is bounded.

Let $c \in]0, 1[$. There exists $r_1 \in \mathbb{R}_+$ so that

$$|f(\mathbf{x})| < \frac{c}{\|g\|_\infty + 1} \quad \forall \mathbf{x} \in \mathbb{R}^{n+m} \setminus \overline{B}_{\mathbb{R}^n}(0; r_1).$$

Similarly, there exists $r_2 \in \mathbb{R}_+$ so that

$$|g(\mathbf{x})| < \frac{c}{\|f\|_\infty + 1} \quad \forall \mathbf{x} \in \mathbb{R}^{n+m} \setminus \overline{B}_{\mathbb{R}^m}(0; r_2).$$

Let $r := 2 \max\{r_1, r_2\}$. Suppose that $\mathbf{y} \in \mathbb{R}^{n+m} \setminus \overline{B}_{\mathbb{R}^n}(0; r)$. Now

$$\|\mathbf{y}\| = \sqrt{\sum_{k=1}^{n+m} (\mathbf{y}[k])^2} = \sqrt{\|A_1 \mathbf{y}\|^2 + \|A_2 \mathbf{y}\|^2} > r.$$

Consequently

$$\begin{aligned}\|A_1 \mathbf{y}\|^2 + \|A_2 \mathbf{y}\|^2 > r^2 &\implies \max\{\|A_1 \mathbf{y}\|^2, \|A_2 \mathbf{y}\|^2\} > \frac{r^2}{2} \\ &\implies \|A_1 \mathbf{y}\| > \max\{r_1, r_2\} \geq r_1 \vee \|A_2 \mathbf{y}\| > \max\{r_1, r_2\} \geq r_2.\end{aligned}$$

In case $\|A_1 \mathbf{y}\| > r_1$

$$|(f \otimes g)(\mathbf{y})| = |f(A_1 \mathbf{y})| |g(A_2 \mathbf{y})| \leq \frac{c}{\|g\|_\infty + 1} \|g\|_\infty < c$$

and in case $\|A_2 \mathbf{y}\| > r_2$

$$|(f \otimes g)(\mathbf{y})| = |f(A_1 \mathbf{y})| |g(A_2 \mathbf{y})| \leq \|f\|_\infty \frac{c}{\|f\|_\infty + 1} < c.$$

Hence $|(f \otimes g)(\mathbf{y})| < c$. Consequently

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} (f \otimes g)(\mathbf{y}) = 0$$

and $f \otimes g \in C_0(\mathbb{R}^{n+m}, K)$. □

Lemma 4.12. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $n, m \in \mathbb{Z}_+$, $f \in C_u(\mathbb{R}^n, K)$ and $g \in C_u(\mathbb{R}^m, K)$. Then $f \otimes g \in C_u(\mathbb{R}^{n+m}, K)$.*

Lemma 4.13. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Then*

$$\bigotimes_{j=1}^n \hat{\otimes}_\varepsilon C_u(\mathbb{R}, K) \subset_{c.s.} C_u(\mathbb{R}^n, K)$$

Proof. Define

$$F^{[n']} :=_{\text{n.s.}} \bigotimes_{j=1}^{n'} \hat{\otimes}_\varepsilon C_u(\mathbb{R}, K)$$

for all $n' \in \mathbb{Z}$. By Corollary 4.10 $F^{[n']} \subset_{c.s.} C_b(\mathbb{R}^n, K)$ for each $n' \in \mathbb{Z}_+$. When $n = 1$ the lemma is true. Suppose that the lemma is true for some $n' \in \mathbb{Z}_+$. By Lemma 4.12 $F^{[n']} \otimes_\varepsilon C_u(\mathbb{R}, K) \subset_{\text{n.s.}} C_u(\mathbb{R}^{n'+1}, K)$. As $C_u(\mathbb{R}^{n'+1}, K)$ is complete it follows that $F^{[n'+1]} =_{\text{n.s.}} F^{[n']} \hat{\otimes}_\varepsilon C_u(\mathbb{R}, K) \subset_{c.s.} C_u(\mathbb{R}^{n'+1}, K)$. □

Lemma 4.14. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Then*

$$\bigotimes_{j=1}^n \hat{\otimes}_\varepsilon C_0(\mathbb{R}, K) \subset_{c.s.} C_0(\mathbb{R}^n, K).$$

Proof. Define

$$F^{[n']} :=_{\text{n.s.}} \bigotimes_{j=1}^{n'} \hat{\otimes}_\varepsilon C_0(\mathbb{R}, K)$$

for all $n' \in \mathbb{Z}$. By Corollary 4.10 $F^{[n']} \subset_{c.s.} C_b(\mathbb{R}^n, K)$ for each $n' \in \mathbb{Z}_+$. When $n = 1$ the lemma is true. Suppose that the lemma is true for some $n' \in \mathbb{Z}_+$. By Lemma 4.11 $F^{[n']} \otimes_\varepsilon C_0(\mathbb{R}, K) \subset_{\text{n.s.}} C_0(\mathbb{R}^{n'+1}, K)$. As $C_0(\mathbb{R}^{n'+1}, K)$ is complete it follows that $F^{[n'+1]} =_{\text{n.s.}} F^{[n']} \hat{\otimes}_\varepsilon C_0(\mathbb{R}, K) \subset_{c.s.} C_0(\mathbb{R}^{n'+1}, K)$. □

Lemma 4.15. Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Let α be a uniform crossnorm. Let E_1, E_2, F_1 , and F_2 be Banach spaces. Let A_1 be a closed subspace of E_1 , A_2 a closed subspace of E_2 , B_1 a closed subspace of F_1 , and B_2 a closed subspace of F_2 . Suppose that A_1 is topologically complemented in E_1 and A_2 is topologically complemented in E_2 . Let $S : A_1 \rightarrow B_1$ and $T : A_2 \rightarrow B_2$ be operators. Then $S \otimes T$ is an operator from $A_1 \otimes_{\alpha; E_1 \otimes E_2} A_2$ into $B_1 \otimes_{\alpha; F_1 \otimes F_2} B_2$.

Proof. Let P_1 be a continuous projection of E_1 onto A_1 and P_2 be a continuous projection of E_2 onto A_2 . Let $X = E_1 \otimes_{\alpha} E_2$ and $Y = F_1 \otimes_{\alpha} F_2$. Let $R = (S \circ P_1) \otimes (T \circ P_2)$. Now R is an operator from X into Y and $R[X] = (S \otimes T)[A_1 \otimes_{\alpha; E_1 \otimes E_2} A_2] \subset B_1 \otimes_{\alpha; F_1 \otimes F_2} B_2$. Hence $S \otimes T = R|_{(A_1 \otimes_{\alpha; E_1 \otimes E_2} A_2)}$ is an operator and

$$\|S \otimes T\| \leq \|R\| = \|S \circ P_1\| \|T \circ P_2\| \leq \|S\| \|T\| \|P_1\| \|P_2\|.$$

□

Definition 4.16. Let E_1, E_2 , and F Banach spaces so that $E_1 \otimes E_2$ is a linear subspace of F . Define $E_1 \otimes_{(F)} E_2$ to be the vector space $E_1 \otimes E_2$ equipped with the norm inherited from F and

$$E_1 \hat{\otimes}_{(F)} E_2 :=_{\text{n.s.}} \text{clos}_F (E_1 \otimes_{(F)} E_2).$$

Definition 4.17. Let $n \in \mathbb{Z}_+$. Let A_1, \dots, A_n and B_1, \dots, B_n be Banach spaces so that A_1 is a closed subspace of B_1 and $B_k \otimes A_{k+1}$ is a linear subspace of B_{k+1} for $k \in Z(n-1)$. When $k \geq 2$ define

$$\bigotimes_{j=1}^k_{(B_j)} A_j :=_{\text{n.s.}} \text{clos}_{B_k} T_k$$

where

$$T_k :=_{\text{n.s.}} \left(\bigotimes_{j=1}^{k-1}_{(B_j)} A_j \right) \otimes_{(B_k)} A_k.$$

When $k = 1$ define

$$\bigotimes_{j=1}^k_{(B_j)} A_j :=_{\text{n.s.}} A_1.$$

When the assumptions of Definition 4.17 hold

$$\bigotimes_{j=1}^k_{(B_j)} A_j$$

is a closed subspace of Banach space B_k for all $k \in Z(n)$.

Definition 4.18. Let A_1, A_2, B_1, B_2, E , and F be Banach spaces so that $A_1 \otimes A_2$ is a linear subspace of E and $B_1 \otimes B_2$ is a linear subspace of F . Let $P_1 : A_1 \rightarrow B_1$ and $P_2 : A_2 \rightarrow B_2$ be operators so that $P_1 \otimes P_2$ is an operator from $A_1 \otimes_{(E)} A_2$ into $B_1 \otimes_{(F)} B_2$. Define operator $P_1 \otimes_{(E, F)} P_2 : A_1 \hat{\otimes}_{(E)} A_2 \rightarrow B_1 \hat{\otimes}_{(F)} B_2$ to be the unique continuous linear extension of $P_1 \otimes P_2$ to $A_1 \hat{\otimes}_{(E)} A_2$.

The extension in the previous definition always exists. See [26, theorem 3.4.4] and [38, chapter 6.1].

Definition 4.19. Let $n \in \mathbb{Z}_+$. Let $A_1, \dots, A_n, B_1, \dots, B_n, E_1, \dots, E_n$, and F_1, \dots, F_n be Banach spaces so that

- A_1 is a closed subspace of E_1 and $E_k \otimes A_{k+1}$ is a linear subspace of E_{k+1} for all $k = 1, \dots, n-1$.
- B_1 is a closed subspace of F_1 and $F_k \otimes B_{k+1}$ is a linear subspace of F_{k+1} for all $k = 1, \dots, n-1$.

Suppose that $P_k : A_k \rightarrow B_k$, $k = 1, \dots, n$ are operators. Let $S_1 := P_1$, $T_1 := S_1 = P_1$, and $S_k := T_{k-1} \otimes P_k$ for $k = 2, \dots, n$. When $k \in \{2, \dots, n\}$ and S_k is continuous let

$$T_k : \bigotimes_{l=1}^k \underset{(E_l)}{\hat{\otimes}} A_l \rightarrow \bigotimes_{l=1}^k \underset{(F_l)}{\hat{\otimes}} B_l,$$

be the unique continuous linear extension of S_k to

$$\bigotimes_{l=1}^k \underset{(E_l)}{\hat{\otimes}} A_l.$$

If $k \in \{2, \dots, n\}$ and S_k is not continuous let $T_k = 0$. When all of the functions S_k , T_k , $k = 1, \dots, n$ are operators define

$$\bigotimes_{k=1}^n \underset{(E_k, F_k)}{\hat{\otimes}} P_k = T_n.$$

If any of the functions S_k , T_k , $k = 1, \dots, n$ is not an operator then

$$\bigotimes_{k=1}^n \underset{(E_k, F_k)}{\hat{\otimes}} P_k$$

is undefined.

Lemma 4.20. *Let $n \in \mathbb{Z}_+$. Let A_1, \dots, A_n and B_1, \dots, B_n be Banach spaces so that B_k is a closed subspace of A_k for each $k \in Z(n)$. Let α be a uniform crossnorm. Let $P_k : A_k \rightarrow B_k$ be a continuous projection of A_k onto B_k for each $k \in Z(n)$. Let*

$$E_n :=_{\text{n.s.}} \bigotimes_{k=1}^n \underset{\alpha}{\hat{\otimes}} A_k, \quad F_n :=_{\text{n.s.}} \bigotimes_{k=1}^n \underset{\alpha}{\hat{\otimes}} B_k, \quad \text{and} \quad Q_n :=_{\text{n.s.}} \bigotimes_{k=1}^n \underset{\alpha}{\hat{\otimes}} P_k.$$

Suppose that $F_n \subset_{\text{c.s.}} E_n$. Then Q_n is a continuous projection of E_n onto F_n .

Proof. Q_n is a continuous linear function from E_n into F_n . We shall prove by induction that Q_n is a projection onto F_n . Suppose that the proposition is true for some $n' \in Z(n-1)$ (induction assumption). Then $Q_{n'+1} = Q_{n'} \otimes_{\alpha} P_{n'+1}$, $E_{n'+1} = E_{n'} \hat{\otimes}_{\alpha} A_{n'+1}$, and $F_{n'+1} = F_{n'} \hat{\otimes}_{\alpha} B_{n'+1}$. Let $x \in F_{n'+1}$. There exists $(x_k)_{k=0}^{\infty} \subset F_{n'} \otimes_{\alpha} B_{n'+1}$ so that $x_k \rightarrow x$ as $k \rightarrow \infty$. Let $k_1 \in \mathbb{N}$. Then

$$x_{k_1} = \sum_{j=1}^m f_j \otimes b_j$$

where $m \in \mathbb{Z}_+$ and $f_j \in F_{n'}$, $b_j \in B_{n'+1}$ for each $j \in Z(m)$. Now

$$Q_{n'+1} x_{k_1} = \sum_{j=1}^m (Q_{n'} f_j) \otimes (P_{n'+1} b_j) = \sum_{j=1}^m f_j \otimes b_j = x_{k_1}$$

where the second equality follows the induction assumption. Since $k_1 \in \mathbb{N}$ was arbitrary and $Q_{n'+1}$ is continuous $Q_{n'+1} x = x$. Therefore the proposition is true for $n' + 1$ and hence for all $n \in \mathbb{Z}_+$. \square

Uniform crossnorms behave well with isometric and topological isomorphisms. The following two lemmas are related to this.

Lemma 4.21. *Let α be a uniform crossnorm. Let A, B, A' , and B' be Banach spaces so that $A \cong_1 A'$ and $B \cong_1 B'$. Let $\iota_A : A \rightarrow A'$ and $\iota_B : B \rightarrow B'$ be isometric isomorphisms. Then $\iota_A \otimes_\alpha \iota_B$ is an isometric isomorphism from $A \hat{\otimes}_\alpha B$ onto $A' \hat{\otimes}_\alpha B'$.*

Proof. Function $\iota_A \otimes_\alpha \iota_B$ is a linear and continuous function from $A \hat{\otimes}_\alpha B$ into $A' \hat{\otimes}_\alpha B'$.

We prove first that $\iota_A \otimes_\alpha \iota_B$ is a surjection onto $A' \hat{\otimes}_\alpha B'$. Let $v \in A' \hat{\otimes}_\alpha B'$. There exists a sequence $(v_k)_{k=0}^\infty \subset A' \otimes_\alpha B'$ so that $v_k \rightarrow v$ as $k \rightarrow \infty$. Now

$$v_k = \sum_{j=1}^{m_k} x_{k,j} \otimes y_{k,j} \quad \forall k \in \mathbb{N}$$

where $m_k \in \mathbb{Z}_+$ for all $k \in \mathbb{N}$ and $x_{k,j} \in A'$, $y_{k,j} \in B'$ for all $k \in \mathbb{Z}_+$ and $j \in Z(m_k)$. Let $a_{k,j} = \iota_A^{-1}(x_{k,j})$ and $b_{k,j} = \iota_B^{-1}(y_{k,j})$ for all $k \in \mathbb{Z}_+$ and $j \in Z(m_k)$. Define

$$u_k := \sum_{j=1}^{m_k} a_{k,j} \otimes b_{k,j} \quad \forall k \in \mathbb{N}$$

Now $(\iota_A \otimes_\alpha \iota_B)(u_k) = v_k$ and $(\iota_A^{-1} \otimes_\alpha \iota_B^{-1})(v_k) = u_k$ for all $k \in \mathbb{N}$. Since ι_A^{-1} is an isometric isomorphism from A' onto A and ι_B^{-1} is an isometric isomorphism from B' onto B it follows that $\iota_A^{-1} \otimes_\alpha \iota_B^{-1}$ is a continuous linear function from $A' \hat{\otimes}_\alpha B'$ into $A \hat{\otimes}_\alpha B$. Therefore $u_k = (\iota_A^{-1} \otimes_\alpha \iota_B^{-1})(v_k) \rightarrow (\iota_A^{-1} \otimes_\alpha \iota_B^{-1})(v) \in A \hat{\otimes}_\alpha B$ as $k \rightarrow \infty$. Furthermore, $(\iota_A \otimes_\alpha \iota_B)(u_k) = v_k \rightarrow v$ as $k \rightarrow \infty$ and it follows that $(\iota_A \otimes_\alpha \iota_B)((\iota_A^{-1} \otimes_\alpha \iota_B^{-1})(v)) = v$. Hence $\iota_A \otimes_\alpha \iota_B$ is a surjection onto $A' \hat{\otimes}_\alpha B'$.

We then prove that $\iota_A \otimes_\alpha \iota_B$ is distance preserving. Let $v \in A \hat{\otimes}_\alpha B$. Since α is a uniform crossnorm it follows that $\|\iota_A \otimes_\alpha \iota_B\| \leq \|\iota_A\| \|\iota_B\| = 1$ and $\|\iota_A^{-1} \otimes_\alpha \iota_B^{-1}\| \leq \|\iota_A^{-1}\| \|\iota_B^{-1}\| = 1$. Now

$$\alpha_{A',B'}((\iota_A \otimes_\alpha \iota_B)(v)) \leq \|\iota_A \otimes_\alpha \iota_B\| \alpha_{A,B}(v) \leq \alpha_{A,B}(v)$$

and

$$\begin{aligned} \alpha_{A,B}(v) &= \alpha_{A,B}((\iota_A^{-1} \otimes_\alpha \iota_B^{-1})((\iota_A \otimes_\alpha \iota_B)(v))) \leq \|\iota_A^{-1} \otimes_\alpha \iota_B^{-1}\| \alpha_{A',B'}((\iota_A \otimes_\alpha \iota_B)(v)) \\ &\leq \alpha_{A',B'}((\iota_A \otimes_\alpha \iota_B)(v)). \end{aligned}$$

Consequently $\alpha_{A,B}(v) = \alpha_{A',B'}((\iota_A \otimes_\alpha \iota_B)(v))$. Thus function $\iota_A \otimes_\alpha \iota_B$ is distance preserving and an isometric isomorphism from $A \hat{\otimes}_\alpha B$ onto $A' \hat{\otimes}_\alpha B'$. \square

Lemma 4.22. *Let α be a uniform crossnorm. Let A, B, A' , and B' be Banach spaces. Let $\iota_A : A \rightarrow A'$ and $\iota_B : B \rightarrow B'$ be topological isomorphisms. Then $\iota_A \otimes_\alpha \iota_B$ is a topological isomorphism from $A \hat{\otimes}_\alpha B$ onto $A' \hat{\otimes}_\alpha B'$.*

Proof. Function $\iota_A \otimes_\alpha \iota_B$ is a linear and continuous function from $A \hat{\otimes}_\alpha B$ into $A' \hat{\otimes}_\alpha B'$. The proof of the surjectivity of $\iota_A \otimes_\alpha \iota_B$ is similar to that in the proof of Lemma 4.21.

We then prove that $\iota_A \otimes_\alpha \iota_B$ defines a norm equivalence between spaces $A \hat{\otimes}_\alpha B$ and $A' \hat{\otimes}_\alpha B'$. We may assume that none of the spaces A , B , A' , or B' is $\{0\}$. We have $\|\iota_A \otimes_\alpha \iota_B\| > 0$ and $\|\iota_A^{-1} \otimes_\alpha \iota_B^{-1}\| > 0$ as neither of these two mappings is identically zero. Now $\alpha_{A',B'}((\iota_A \otimes_\alpha \iota_B)(v)) \leq \|\iota_A \otimes_\alpha \iota_B\| \alpha_{A,B}(v)$ and $\alpha_{A,B}(v) = \alpha_{A,B}((\iota_A^{-1} \otimes_\alpha \iota_B^{-1})((\iota_A \otimes_\alpha \iota_B)(v))) \leq \|\iota_A^{-1} \otimes_\alpha \iota_B^{-1}\| \alpha_{A',B'}((\iota_A \otimes_\alpha \iota_B)(v))$ from which it follows that

$$\frac{1}{\|\iota_A^{-1} \otimes_\alpha \iota_B^{-1}\|} \alpha_{A,B}(v) \leq \alpha_{A',B'}((\iota_A \otimes_\alpha \iota_B)(v)).$$

Consequently

$$\frac{1}{\|\iota_A^{-1} \otimes_{\alpha} \iota_B^{-1}\|} \alpha_{A,B}(v) \leq \alpha_{A',B'}((\iota_A \otimes_{\alpha} \iota_B)(v)) \leq \|\iota_A^{-1} \otimes_{\alpha} \iota_B^{-1}\| \alpha_{A',B'}((\iota_A \otimes_{\alpha} \iota_B)(v)).$$

Hence function $\iota_A \otimes_{\alpha} \iota_B$ is defines a norm equivalence between spaces $A \hat{\otimes}_{\alpha} B$ and $A' \hat{\otimes}_{\alpha} B'$. If $x, y \in A \hat{\otimes}_{\alpha} B$, $x \neq y$ it follows that

$$\begin{aligned} \alpha_{A',B'}((\iota_A \otimes_{\alpha} \iota_B)(x) - (\iota_A \otimes_{\alpha} \iota_B)(y)) &= \alpha_{A',B'}((\iota_A \otimes_{\alpha} \iota_B)(x - y)) \\ &\geq \frac{1}{\|\iota_A^{-1} \otimes_{\alpha} \iota_B^{-1}\|} \alpha_{A,B}(x - y) > 0 \end{aligned}$$

and consequently $(\iota_A \otimes_{\alpha} \iota_B)(x) \neq (\iota_A \otimes_{\alpha} \iota_B)(y)$. Hence function $\iota_A \otimes_{\alpha} \iota_B$ is a topological isomorphism from $A \hat{\otimes}_{\alpha} B$ onto $A' \hat{\otimes}_{\alpha} B'$. \square

Book [38] has an exercise for proving isometric isomorphism $l^1 \hat{\otimes}_{\pi} l^1 \cong_1 l^1$. However, we also need to construct the isometric isomorphism for this article so we give the following lemma without proof.

Lemma 4.23. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. Define function*

$$\iota : l^1(\mathbb{N}, K) \rightarrow l^1(\mathbb{N}, K) \hat{\otimes}_{\pi} l^1(\mathbb{N}, K)$$

by

$$\iota(\mathbf{e}_k) := \mathbf{e}_{\sigma_{\text{sq}}(k)}^{\otimes}, \quad k \in \mathbb{N},$$

and extending by linearity and continuity onto whole $l^1(\mathbb{N}, K)$. Then ι is an isometric isomorphism from Banach space $l^1(\mathbb{N}, K)$ onto Banach space $l^1(\mathbb{N}, K) \hat{\otimes}_{\pi} l^1(\mathbb{N}, K)$.

This result can be generalized for the completed projective tensor product of more than two $l^1(\mathbb{N}, K)$:

Lemma 4.24. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Define function*

$$\eta^{[n]} : l^1(\mathbb{N}, K) \rightarrow \bigotimes_{k=1}^n \pi l^1(\mathbb{N}, K)$$

by

$$\eta^{[n]}(\mathbf{e}_k) := \mathbf{e}_{\sigma_{\text{sq}}^{[n]}(k)}^{\otimes}, \quad k \in \mathbb{N},$$

and extending by linearity and continuity onto whole $l^1(\mathbb{N}, K)$. Then η is an isometric isomorphism from Banach space $l^1(\mathbb{N}, K)$ onto Banach space

$$\bigotimes_{k=1}^n \pi l^1(\mathbb{N}, K).$$

Proof. Case $n = 1$ is trivial and case $n = 2$ is true by Lemma 4.23. The lemma can be proven using induction by n and equation

$$\forall n \in \mathbb{N} + 2 : \eta^{[n+1]} = (\eta^{[n]} \otimes_{\pi} \text{id}_{l^1(\mathbb{N}, K)}) \circ \eta^{[2]}.$$

\square

Lemma 4.25. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Define function*

$$\xi^{[n]} : \bigotimes_{k=1}^n l^1(\mathbb{Z}, K) \rightarrow l^1(\mathbb{Z}^n, K)$$

by

$$\xi^{[n]}(\check{\mathbf{e}}_{\mathbf{k}}) := \check{\mathbf{e}}_{\mathbf{k}}, \quad \mathbf{k} \in \mathbb{Z}^n,$$

and extending by linearity and continuity onto whole $\bigotimes_{k=1}^n l^1(\mathbb{Z}, K)$. Then $\xi^{[n]}$ is well-defined and it is an isometric isomorphism from Banach space $\bigotimes_{k=1}^n l^1(\mathbb{Z}, K)$ onto Banach space $l^1(\mathbb{Z}^n, K)$.

Proof. Let α be some bijection from \mathbb{Z} onto \mathbb{N} . Define function $\beta : l^1(\mathbb{Z}, K) \rightarrow l^1(\mathbb{N}, K)$ by

$$\beta(\check{\mathbf{e}}_k) := \mathbf{e}_{\alpha(k)}$$

for all $k \in \mathbb{Z}$. and extending by linearity and continuity onto whole $l^1(\mathbb{Z}, K)$. Function β is an isometric isomorphism from Banach space $l^1(\mathbb{Z}, K)$ onto Banach space $l^1(\mathbb{N}, K)$. Define

$$\beta^{[n]} := \bigotimes_{k=1}^n \beta.$$

Now $\beta^{[n]}$ is an isometric isomorphism from Banach space

$$\bigotimes_{k=1}^n l^1(\mathbb{Z}, K)$$

onto Banach space

$$\bigotimes_{k=1}^n l^1(\mathbb{N}, K).$$

Let

$$\alpha^{[n]}(\ell) := \sum_{k=1}^n \alpha(\ell[k]) \mathbf{e}_k^{[n]}$$

for all $\ell \in \mathbb{Z}^n$. It can be shown that $\alpha^{[n]}$ is a bijection from \mathbb{Z}^n onto \mathbb{N}^n . Let $\rho := (\alpha^{[n]})^{-1} \circ \sigma_{\text{sq}}^{[n]}$. Now ρ is a bijection from \mathbb{N} onto \mathbb{Z}^n . Define function $\gamma : l^1(\mathbb{N}, K) \rightarrow l^1(\mathbb{Z}^n, K)$ by

$$\gamma(\mathbf{e}_k) := \check{\mathbf{e}}_{\rho(k)}, \quad k \in \mathbb{N},$$

and extending by linearity and continuity onto whole $l^1(\mathbb{N}, K)$. Function γ is an isometric isomorphism from Banach space $l^1(\mathbb{N}, K)$ onto Banach space $l^1(\mathbb{Z}^n, K)$. Define function $\eta^{[n]}$ as in Lemma 4.24. Let $\xi^{[n]} := \gamma \circ (\eta^{[n]})^{-1} \circ \beta^{[n]}$ Now function $\xi^{[n]}$ is an isometric isomorphism from Banach space

$$\bigotimes_{k=1}^n l^1(\mathbb{Z}, K)$$

onto Banach space $l^1(\mathbb{Z}^n, K)$. □

We give then a lemma about multiplication of series in spaces topologically isomorphic to c_0 or l^1 .

Lemma 4.26. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let F be a Banach space with scalar field K and α be a uniform crossnorm. Let $X =_{\text{n.s.}} c_0(\mathbb{N}, K)$ or $X =_{\text{n.s.}} l^1(\mathbb{N}, K)$. Let $\eta : F \rightarrow X$ be a topological isomorphism and suppose that $\eta(f_j) = \mathbf{e}_j$ where $f_j \in F$ for each $j \in \mathbb{N}$. Let $\mathbf{a}_k \in X$ for all $k \in \mathbb{Z}(n)$. Then*

$$\bigotimes_{l=1}^n \left(\sum_{j \in \mathbb{N}} \mathbf{a}_l[j] f_j \right) = \sum_{\mathbf{k} \in \mathbb{N}^n} \left(\prod_{l=1}^n \mathbf{a}_l[\mathbf{k}[l]] \right) f_{\mathbf{k}[1]} \otimes \cdots \otimes f_{\mathbf{k}[n]} = \sum_{\mathbf{k} \in \mathbb{N}^n} \bigotimes_{l=1}^n \mathbf{a}_l[\mathbf{k}[l]] f_{\mathbf{k}[l]}$$

where the second and third series are computed on space $\bigotimes_{\alpha, k=1}^n F$.

Proof. Let

$$\eta^{[n]} := \bigotimes_{\alpha, k=1}^n \eta, \quad F^{[n]} :=_{\text{n.s.}} \bigotimes_{\alpha, k=1}^n F, \quad \text{and} \quad X^{[n]} :=_{\text{n.s.}} \bigotimes_{\alpha, k=1}^n X.$$

By Lemma 4.22 $\eta^{[n]}$ is a topological isomorphism from $F^{[n]}$ onto $X^{[n]}$.

Let

$$t := \bigotimes_{k=1}^n \left(\sum_{j \in \mathbb{N}} \mathbf{a}_k[j] f_j \right).$$

Now

$$\begin{aligned} \eta^{[n]}(t) &= \bigotimes_{l=1}^n \left(\eta \left(\sum_{j \in \mathbb{N}} \mathbf{a}_l[j] f_j \right) \right) = \bigotimes_{l=1}^n \left(\sum_{j \in \mathbb{N}} \mathbf{a}_l[j] \mathbf{e}_j \right) \\ &= \sum_{m_1 \in \mathbb{N}} \cdots \sum_{m_n \in \mathbb{N}} \left(\prod_{l=1}^n \mathbf{a}_l[m_l] \right) \mathbf{e}_{m_1} \otimes \cdots \otimes \mathbf{e}_{m_n} \end{aligned}$$

and

$$\left(\eta^{[n]}(t) \right) [\mathbf{k}] = \prod_{l=1}^n \mathbf{a}_l[\mathbf{k}[l]]$$

for each $\mathbf{k} \in \mathbb{N}^n$. Consequently

$$\eta^{[n]}(t) = \sum_{\mathbf{k} \in \mathbb{N}^n} \left(\prod_{l=1}^n \mathbf{a}_l[\mathbf{k}[l]] \right) \mathbf{e}_{\mathbf{k}[1]} \otimes \cdots \otimes \mathbf{e}_{\mathbf{k}[n]}.$$

It follows that

$$t = \left(\eta^{[n]} \right)^{-1} \left(\eta^{[n]}(t) \right) = \sum_{\lambda \in \mathbb{N}^n} \left(\prod_{l=1}^n \mathbf{a}_l[\lambda[l]] \right) f_{\lambda[1]} \otimes \cdots \otimes f_{\lambda[n]} = \sum_{\mathbf{k} \in \mathbb{N}^n} \bigotimes_{l=1}^n \mathbf{a}_l[\mathbf{k}[l]] f_{\mathbf{k}[l]}.$$

□

5 General Definitions for a Compactly Supported Interpolating MRA

5.1 Mother Scaling Function

Definition 5.1. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. A *compactly supported interpolating mother scaling function* is a function $\varphi \in C_{\text{com}}(\mathbb{R}^n, K)$ satisfying the following conditions:

(MSF.1)

$$\forall \mathbf{k} \in \mathbb{Z}^n : \varphi(\mathbf{k}) = \delta_{\mathbf{k},0}$$

(MSF.2)

$$\forall \mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \varphi\left(\frac{\mathbf{k}}{2}\right) \varphi(2\mathbf{x} - \mathbf{k}).$$

5.2 General Definitions for the Univariate MRA's

We shall denote the function space for which the MRA is defined by E in this section. We have either $E = C_u(\mathbb{R}, K)$ or $E = C_0(\mathbb{R}, K)$ where $K = \mathbb{R}$ or $K = \mathbb{C}$. We shall assume that $\varphi \in C_{\text{com}}(\mathbb{R}, K)$ is a compactly supported interpolating mother scaling function throughout this subsection.

Definition 5.2. Define function $\psi \in C_{\text{com}}(\mathbb{R}, K)$ by

$$\psi(x) := \varphi(2x - 1)$$

for all $x \in \mathbb{R}$. Function ψ is called *the mother wavelet*.

Definition 5.3. Define

$$\varphi_{j,k} := \varphi(2^j \cdot -k)$$

and

$$\psi_{j,k} := \psi(2^j \cdot -k)$$

for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

Definition 5.4. Define

$$\zeta_s := \begin{cases} \varphi; & s = 0 \\ \psi; & s = 1 \end{cases}$$

and

$$\zeta_{s,j,k} := \begin{cases} \varphi_{j,k}; & s = 0 \\ \psi_{j,k}; & s = 1 \end{cases}$$

where $s \in \{0, 1\}$, $j \in \mathbb{Z}$, and $k \in \mathbb{Z}$.

Definition 5.5. When $k \in \mathbb{Z}$ define

$$\begin{aligned} h_k &:= \varphi\left(\frac{k}{2}\right) \\ g_k &:= \delta_{k,1} \\ \tilde{h}_k &:= \delta_{k,0} \\ \tilde{g}_k &:= (-1)^{k-1} h_{1-k}. \end{aligned}$$

Definition 5.6. When $t \in \{0, 1\}$ and $k \in \mathbb{Z}$ define

$$g_{t,k} := \begin{cases} h_k; & t = 0 \\ g_k; & t = 1 \end{cases}$$

and

$$\tilde{g}_{t,k} := \begin{cases} \tilde{h}_k; & t = 0 \\ \tilde{g}_k; & t = 1 \end{cases}.$$

Definition 5.7. Define $\tilde{\varphi} := \delta \in E^*$ and $\tilde{\psi} \in E^*$ by

$$\tilde{\psi} := 2 \sum_{k \in \mathbb{Z}} \tilde{g}_k \tilde{\varphi}(2 \cdot -k). \quad (26)$$

Define $\tilde{\varphi}_{j,k} := 2^j \tilde{\varphi}(2^j \cdot -k)$ and $\tilde{\psi}_{j,k} := 2^j \tilde{\psi}(2^j \cdot -k)$ where $j, k \in \mathbb{Z}$.

As φ is compactly supported only a finite number of numbers h_k , $k \in \mathbb{Z}$, and the other three filters defined by Definition 5.5 are nonzero. Consequently the series in Equation (26) has a finite number of nonzero terms.

Definition 5.8. Define

$$\tilde{\zeta}_s := \begin{cases} \tilde{\varphi}; & s = 0 \\ \tilde{\psi}; & s = 1 \end{cases}$$

and

$$\tilde{\zeta}_{s,j,k} := \begin{cases} \tilde{\varphi}_{j,k}; & s = 0 \\ \tilde{\psi}_{j,k}; & s = 1 \end{cases}$$

where $j, k \in \mathbb{Z}$ and $s \in \{0, 1\}$.

Lemma 5.9. Let $k, l \in \mathbb{Z}$. Then

- (i) $\langle \tilde{\varphi}_{j,k}, \varphi_{j,l} \rangle = \delta_{k,l}$
- (ii) $\langle \tilde{\varphi}_{j,k}, \psi_{j,l} \rangle = 0$
- (iii) $\langle \tilde{\psi}_{j,k}, \varphi_{j,l} \rangle = 0$
- (iv) $\langle \tilde{\psi}_{j,k}, \psi_{j,l} \rangle = \delta_{k,l}$

Proof. See also [7, section 2] and [19].

- (i) $\langle \delta(2^j \cdot -k), \varphi(2^j \cdot -l) \rangle = \langle \frac{1}{2^j} \delta \left(\cdot - \frac{k}{2^j} \right), \varphi(2^j \cdot -l) \rangle = \frac{1}{2^j} \varphi(k-l) = \frac{1}{2^j} \delta_{k,l}$

The last equality follows from (MSF.1).

- (ii) As $2k - 2l - 1 \neq 0$ it follows from (MSF.1) that

$$\langle \delta(2^j \cdot -k), \psi(2^j \cdot -l) \rangle = \frac{1}{2^j} \psi(k-l) = \frac{1}{2^j} \varphi(2k - 2l - 1) = 0.$$

(iii)

$$\begin{aligned}
\langle \tilde{\psi}(2^j \cdot -k), \varphi(2^j \cdot -l) \rangle &= 2 \sum_{\nu \in \mathbb{Z}} \tilde{g}_\nu \frac{1}{2^{j+1}} \langle \delta(\cdot - \frac{2k+\nu}{2^{j+1}}), \varphi(2^j \cdot -l) \rangle \\
&= \frac{1}{2^j} \sum_{\nu \in \mathbb{Z}} \tilde{g}_\nu \varphi\left(k - l + \frac{\nu}{2}\right) \\
&= \frac{1}{2^j} \sum_{\nu \in \mathbb{Z}} \tilde{g}_\nu \sum_{\mu \in \mathbb{Z}} \varphi\left(\frac{\mu}{2}\right) \varphi(2k - 2l + \nu - \mu) \\
&= \frac{1}{2^j} \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} (-1)^{\nu-1} \varphi\left(\frac{1-\nu}{2}\right) \varphi\left(\frac{\mu}{2}\right) \delta_{2l-2k, \nu-\mu} \\
&= \frac{1}{2^j} \sum_{\nu \in \mathbb{Z}} (-1)^{\nu-1} \varphi\left(\frac{1-\nu}{2}\right) \varphi\left(\frac{\nu-2l+2k}{2}\right) \\
&= \frac{1}{2^j} \left(\varphi\left(\frac{1-2l+2k}{2}\right) - \sum_{\nu \in \mathbb{Z}} \varphi\left(\frac{1-2\nu}{2}\right) \varphi(2\nu' - 2l + 2k) \right) \\
&= \frac{1}{2^j} \left(\varphi\left(\frac{1-2l+2k}{2}\right) - \varphi\left(\frac{1-2l+2k}{2}\right) \right) = 0
\end{aligned}$$

The 6th equality follows from (MSF.1).

(iv)

$$\begin{aligned}
\langle \tilde{\psi}(2^j \cdot -k), \psi(2^j \cdot -l) \rangle &= \left\langle 2 \sum_{\nu \in \mathbb{Z}} \tilde{g}_\nu \delta(2^{j+1} \cdot -2k - \nu), \varphi(2^{j+1} \cdot -2l - 1) \right\rangle \\
&= \frac{1}{2^j} \sum_{\nu \in \mathbb{Z}} \tilde{g}_\nu \varphi(2k + \nu - 2l - 1) \\
&= \frac{1}{2^j} \sum_{\nu \in \mathbb{Z}} \tilde{g}_\nu \delta_{\nu, 2l-2k+1} = \frac{1}{2^j} \delta_{k,l}
\end{aligned}$$

The last equality follows from the fact that $\tilde{g}_\nu = 0$ for all $\nu \in \mathbb{Z}$, ν odd, and $\nu \neq 1$.

□

5.3 General Definitions for Multivariate MRA's

We will assume that $n \in \mathbb{Z}_+$ and either $K = \mathbb{R}$ or $K = \mathbb{C}$ throughout this subsection. We will also assume that $E =_{\text{n.s.}} C_u(\mathbb{R}, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}, K)$. Furthermore, $\varphi \in C_{\text{com}}(\mathbb{R}, K)$ shall be a compactly supported interpolating mother scaling function throughout this subsection.

We set

$$F :=_{\text{n.s.}} \begin{cases} C_u(\mathbb{R}^n, K); & E = C_u(\mathbb{R}, K) \\ C_0(\mathbb{R}^n, K); & E = C_0(\mathbb{R}, K) \end{cases}$$

Definition 5.10. Define function $\varphi^{[n]} \in C_{\text{com}}(\mathbb{R}^n, K)$ by

$$\varphi^{[n]} := \bigotimes_{k=1}^n \varphi.$$

Function $\varphi^{[n]}$ is called an n -dimensional tensor product mother scaling function generated by φ . Define also

$$\varphi_{j,\mathbf{k}}^{[n]} := \varphi^{[n]}(2^j \cdot -\mathbf{k})$$

where $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$.

Function $\varphi^{[n]}$ is a compactly supported interpolating mother scaling function on \mathbb{R}^n .

Definition 5.11. Define function $\psi_{\mathbf{s}}^{[n]} \in C_{\text{com}}(\mathbb{R}^n, K)$ by

$$\psi_{\mathbf{s}}^{[n]} := \bigotimes_{k=1}^n \zeta_{\mathbf{s}[k]}$$

for all $\mathbf{s} \in \{0, 1\}^n$. Define functions $\psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \in C_{\text{com}}(\mathbb{R}^n, K)$ by

$$\psi_{\mathbf{s},j,\mathbf{k}}^{[n]} := \bigotimes_{l=1}^n \zeta_{\mathbf{s}[l],j,\mathbf{k}[l]}$$

for all $\mathbf{s} \in \{0, 1\}^n$, $j \in \mathbb{Z}$, and $\mathbf{k} \in \mathbb{Z}^n$.

We have

$$\psi_{\mathbf{s}}^{[n]} \left(\mathbf{k} + \frac{1}{2} \mathbf{s} \right) = \delta_{\mathbf{k},0} \quad (27)$$

for all $\mathbf{s} \in \{0, 1\}^n$ and $\mathbf{k} \in \mathbb{Z}^n$.

Definition 5.12. Let $n \in \mathbb{Z}_+$, $\mathbf{s} \in \{0, 1\}^n$, and $\mathbf{t} \in \mathbb{Z}^n$. Define

$$g_{\mathbf{s},\mathbf{t}}^{[n]} := \prod_{l=1}^n g_{\mathbf{s}[l],\mathbf{t}[l]}$$

and

$$\tilde{g}_{\mathbf{s},\mathbf{t}}^{[n]} := \prod_{l=1}^n \tilde{g}_{\mathbf{s}[l],\mathbf{t}[l]}.$$

Lemma 5.13. Suppose that $m \in \mathbb{Z}_+$ so that

$$\forall k \in \mathbb{Z} : |k| \geq m \implies h_k = 0.$$

Then

$$\begin{aligned} \psi_{\mathbf{s}}^{[n]} &= \sum_{\ell \in (Z_{\pm}(m))^n} g_{\mathbf{s},\ell}^{[n]} \varphi_{1,\ell}^{[n]} \\ \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} &= \sum_{\ell \in (Z_{\pm}(m))^n} g_{\mathbf{s},\ell}^{[n]} \varphi_{j+1,2\mathbf{k}+\ell}^{[n]} = \sum_{\mathbf{m} \in \mathbb{Z}^n} g_{\mathbf{s},\mathbf{m}-2\mathbf{k}}^{[n]} \varphi_{j+1,\mathbf{m}}^{[n]} \\ \varphi^{[n]} &= \sum_{\ell \in (Z_{\pm}(m))^n} g_{\mathbf{0}_n,\ell}^{[n]} \varphi_{1,\ell}^{[n]} \\ \varphi_{j,\mathbf{k}}^{[n]} &= \sum_{\ell \in (Z_{\pm}(m))^n} g_{\mathbf{0}_n,\ell}^{[n]} \varphi_{j+1,2\mathbf{k}+\ell}^{[n]} = \sum_{\mathbf{m} \in \mathbb{Z}^n} g_{\mathbf{0}_n,\mathbf{m}-2\mathbf{k}}^{[n]} \varphi_{j+1,\mathbf{m}}^{[n]}, \end{aligned}$$

for all $\mathbf{s} \in \{0, 1\}^n$, $j \in \mathbb{Z}$, and $\mathbf{k} \in \mathbb{Z}^n$.

The domain of the Dirac δ functional varies in this article. I.e. we may keep δ as an element of different dual spaces A^* . When $z_1, \dots, z_m \in \mathbb{R}$ we will identify $\delta(\cdot - z_1) \otimes \dots \otimes \delta(\cdot - z_m)$ with $\delta(\cdot - (z_1, \dots, z_m))$.

Definition 5.14. Define $\tilde{\varphi}^{[n]} \in F^*$ by

$$\tilde{\varphi}^{[n]} := \bigotimes_{l=1}^n \tilde{\varphi}$$

and $\tilde{\varphi}_{j,\mathbf{k}}^{[n]} \in F^*$ by

$$\tilde{\varphi}_{j,\mathbf{k}}^{[n]} := \bigotimes_{l=1}^n \tilde{\varphi}_{j,\mathbf{k}[l]}$$

where $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$. Define also $\tilde{\psi}_{\mathbf{s}}^{[n]} \in F^*$ by

$$\tilde{\psi}_{\mathbf{s}}^{[n]} := \bigotimes_{l=1}^n \tilde{\zeta}_{\mathbf{s}[l]}$$

where $\mathbf{s} \in \{0, 1\}^n$ and $\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} \in F^*$ by

$$\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} := \bigotimes_{l=1}^n \tilde{\zeta}_{\mathbf{s}[l],j,\mathbf{k}[l]}$$

where $\mathbf{s} \in \{0, 1\}^n$, $j \in \mathbb{Z}$, and $\mathbf{k} \in \mathbb{Z}^n$.

Note that $\tilde{\varphi}^{[n]} = \delta$ (the Dirac delta functional),

$$\tilde{\varphi}_{j,\mathbf{k}}^{[n]} = 2^{nj} \delta(2^j \cdot -\mathbf{k}) = \delta\left(\cdot - \frac{\mathbf{k}}{2^j}\right),$$

and

$$\tilde{\psi}_{\mathbf{s}}^{[n]} = \sum_{\ell \in \mathbb{Z}^n} \tilde{g}_{\mathbf{s},\ell}^{[n]} \tilde{\varphi}_{1,\ell}^{[n]} \quad (28)$$

$$\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} = 2^{nj} \tilde{\psi}_{\mathbf{s}}^{[n]}(2^j \cdot -\mathbf{k}) = \sum_{\ell \in \mathbb{Z}^n} \tilde{g}_{\mathbf{s},\ell}^{[n]} \delta\left(\cdot - \frac{2\mathbf{k} + \ell}{2^{j+1}}\right) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \tilde{g}_{\mathbf{s},\mathbf{m}-2\mathbf{k}}^{[n]} \tilde{\varphi}_{j+1,\mathbf{m}}^{[n]} \quad (29)$$

for all $j \in \mathbb{Z}$, $\mathbf{s} \in \{0, 1\}^n$, and $\mathbf{k} \in \mathbb{Z}^n$. We also have

$$\left\langle \tilde{\psi}_{\mathbf{s},j,\ell}^{[n]}, \psi_{\mathbf{t},j,\mathbf{k}}^{[n]} \right\rangle = \delta_{\mathbf{s},\mathbf{t}} \delta_{\ell,\mathbf{k}} \quad (30)$$

$$\left\langle \tilde{\varphi}_{j+1,\ell}^{[n]}, \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \right\rangle = \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \left(\frac{\ell}{2^{j+1}} \right) = g_{\mathbf{s},\ell-2\mathbf{k}}^{[n]} \quad (31)$$

$$\left\langle \tilde{\psi}_{\mathbf{s},j,\ell}^{[n]}, \varphi_{j+1,\mathbf{k}}^{[n]} \right\rangle = \tilde{g}_{\mathbf{s},\mathbf{k}-2\ell}^{[n]} \quad (32)$$

for all $j \in \mathbb{Z}$, $\mathbf{k}, \ell \in \mathbb{Z}^n$, and $\mathbf{s}, \mathbf{t} \in \{0, 1\}^n$.

Lemma 5.15. Let $\mathbf{k}, \ell \in \mathbb{Z}^n$. Then

$$\sum_{\mathbf{s} \in \{0,1\}^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} \tilde{g}_{\mathbf{s},\mathbf{k}-2\mathbf{m}}^{[n]} g_{\mathbf{s},\ell-2\mathbf{m}}^{[n]} = \delta_{\mathbf{k},\ell}.$$

Proof. Let

$$a_{i,j,t} := \tilde{g}_{t,\mathbf{k}[i]-2j} g_{t,\ell[i]-2j}.$$

Now

$$\sum_{\mathbf{s} \in \{0,1\}^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} \tilde{g}_{\mathbf{s},\mathbf{k}-2\mathbf{m}}^{[n]} g_{\mathbf{s},\ell-2\mathbf{m}}^{[n]} = \sum_{\mathbf{s} \in \{0,1\}^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} \prod_{i=1}^n a_{i,\mathbf{m}[i],\mathbf{s}[i]} = \prod_{i=1}^n \sum_{j \in \mathbb{Z}} \sum_{t=0}^1 a_{i,j,t}$$

Let

$$b_i := \sum_{j \in \mathbb{Z}} \sum_{t=0}^1 a_{i,j,t}$$

for all $i \in \mathbb{Z}$.

Suppose that $i_0 \in Z(n)$ and $\ell[i_0]$ is even. Then $\ell[i_0] = 2w$ for some $w \in \mathbb{Z}$. It follows that $a_{i_0,j,0} = \delta_{\mathbf{k}[i_0],2j} \delta_{w,j}$ and $a_{i_0,j,1} = \tilde{g}_{\mathbf{k}[i_0]-2j} g_{2w-2j} = 0$. Consequently

$$b_{i_0} = \sum_{j \in \mathbb{Z}} \delta_{\mathbf{k}[i_0],2j} \delta_{w,j} = \delta_{\mathbf{k}[i_0],\ell[i_0]}. \quad (33)$$

Assume then that $i_0 \in Z(n)$ and $\ell[i_0]$ is odd. Now $\ell[i_0] = 2w + 1$ for some $w \in \mathbb{Z}$, $a_{i_0,j,0} = \delta_{\mathbf{k}[i_0],2j} h_{2w+1-2j}$, and $a_{i_0,j,1} = \tilde{g}_{\mathbf{k}[i_0]-2j} g_{2w+1-2j} = (-1)^{\mathbf{k}[i_0]-2w-1} h_{2w+1-\mathbf{k}[i_0]} \delta_{2w+1-2j,1}$. Assume that $\mathbf{k}[i_0]$ is even. Now $\mathbf{k}[i_0] = 2z$ for some $z \in \mathbb{Z}$, $a_{i_0,j,0} = \delta_{z,j} h_{2w+1-2z}$, and $a_{i_0,j,1} = -h_{2w+1-2z} \delta_{w,j}$. It follows that

$$b_{i_0} = \sum_{j \in \mathbb{Z}} \delta_{z,j} h_{2w+1-2z} - \sum_{j \in \mathbb{Z}} h_{2w+1-2z} \delta_{w,j} = 0 = \delta_{\mathbf{k}[i_0],\ell[i_0]}. \quad (34)$$

Assume finally that $\mathbf{k}[i_0]$ is odd. Now $\mathbf{k}[i_0] = 2z + 1$ for some $z \in \mathbb{Z}$. We have $a_{i_0,j,0} = \delta_{2z+1,2l} h_{2w+1-2l} = 0$, $a_{i_0,j,1} = (-1)^{2z+1-2w-1} h_{2w+1-2z-1} \delta_{2w+1-2j,1} = \delta_{w,z} \delta_{w,j}$. It follows that

$$b_{i_0} = \sum_{j \in \mathbb{Z}} a_{i_0,j,0} + \sum_{j \in \mathbb{Z}} a_{i_0,j,1} = \delta_{w,z} = \delta_{\mathbf{k}[i_0],\ell[i_0]}. \quad (35)$$

By Equations (33), (34), and (35) the lemma is true. \square

Goedecker [19] gives also formulas for wavelet filters.

6 Compactly Supported Interpolating MRA of $C_u(\mathbb{R}^n, K)$

We will assume that $n \in \mathbb{Z}_+$ and $K = \mathbb{R}$ or $K = \mathbb{C}$ throughout this section. We will also assume that $\varphi^{[n]} \in C_{\text{com}}(\mathbb{R}^n, K)$ is an n -dimensional tensor product mother scaling function for which $(\tilde{\varphi}^{[n]}, \varphi^{[n]})$ spans all polynomials of degree 0 in this section. Chui and Li [7] have developed a MRA in the univariate case $C_u(\mathbb{R})$.

Definition 6.1. Define

$$V_{n,j}^{(u)} := \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) : \mathbf{a} \in l^\infty(\mathbb{Z}^n, K) \right\}$$

$$\|f|V_{n,j}^{(u)}\| := \|f\|_\infty, \quad f \in V_{n,j}^{(u)}$$

for all $j \in \mathbb{Z}$.

Definition 6.2. Let spaces $V_{n,j}^{(u)}$, $j \in \mathbb{Z}$, be defined by Definition 6.1. We call $\{V_{n,j}^{(u)} : j \in \mathbb{Z}\}$ an *interpolating tensor product MRA* of $C_u(\mathbb{R}^n, K)$ generated by $\varphi^{[n]}$ provided that the following conditions are satisfied:

$$(MRA1.1) \quad \forall j \in \mathbb{Z} : V_{n,j}^{(u)} \subset V_{n,j+1}^{(u)}$$

$$(MRA1.2) \quad \overline{\bigcup_{j \in \mathbb{Z}} V_{n,j}^{(u)}} = C_u(\mathbb{R}^n, K)$$

$$(MRA1.3) \quad \bigcap_{j \in \mathbb{Z}} V_{n,j}^{(u)} = K$$

$$(MRA1.4) \quad \forall j \in \mathbb{Z}, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(u)} \iff f(2 \cdot) \in V_{n,j+1}^{(u)}$$

$$(MRA1.5) \quad \forall j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(u)} \iff f(\cdot - 2^{-j}\mathbf{k}) \in V_{n,j}^{(u)}$$

$$(MRA1.6) \quad \forall \mathbf{k} \in \mathbb{Z}^n : \varphi^{[n]}(\mathbf{k}) = \delta_{\mathbf{k},0}$$

Our requirements for the definition of interpolating multiresolution analysis are stricter and simpler than those in [7]. Condition (MRA1.6) is replaced by a weaker condition for φ in [7] but it is possible to construct function φ_L that satisfies condition (MRA1.6) and generates the same subspaces $V_{1,j}^{(u)}$ as function φ .

Lemma 6.3. *Under the conditions given at the start of this section ($\varphi^{[n]}$ is an n -dimensional tensor product mother scaling function) the conditions (MRA1.1), (MRA1.4), (MRA1.5), and (MRA1.6) are true.*

Proof. Use Lemmas 3.10 and 5.13. □

Define function $\iota_{n,j}^{(u)} : l^\infty(\mathbb{Z}^n, K) \rightarrow V_{n,j}^{(u)}$ by

$$\iota_{n,j}^{(u)}(\mathbf{a}) := \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi_{j,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$. By Lemma 3.10 function $\iota_{n,j}^{(u)}$ is a topological isomorphism from $l^\infty(\mathbb{Z}^n, K)$ onto $V_{n,j}^{(u)}$ and

$$\|\mathbf{a}\|_\infty \leq \left\| \iota_{n,j}^{(u)}(\mathbf{a}) \right\|_\infty \leq N_{\text{cover}}(\varphi^{[n]}) \left\| \varphi^{[n]} \right\|_\infty \|\mathbf{a}\|_\infty \quad (36)$$

for all $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$ and $j \in \mathbb{Z}$.

Theorem 6.4. *We have*

$$\bigcap_{j \in \mathbb{Z}} V_{n,j}^{(u)} = K.$$

Proof. The proof is similar to a part of the proof of [7, theorem 3.2]. Suppose that $g \in V_{n,j}^{(u)}$ for all $j \in \mathbb{Z}$. Now

$$\forall \mathbf{x} \in \mathbb{R}^n, j \in \mathbb{Z} : g(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}_j[\mathbf{k}] \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k})$$

where $\mathbf{a}_j = (a_{j,\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^n} \in l^\infty(\mathbb{Z}^n, K)$ for all $j \in \mathbb{Z}$. It follows from condition (MRA1.6) that

$$\|g\|_\infty \geq \left| g\left(\frac{\mathbf{k}}{2^j}\right) \right| = \left| \sum_{\ell \in \mathbb{Z}^n} a_{j,\ell} \varphi^{[n]}(\mathbf{k} - \ell) \right| = |a_{j,\mathbf{k}}|$$

for all $\mathbf{k} \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$. Hence $\|\mathbf{a}_j\|_\infty \leq \|g\|_\infty$ for all $j \in \mathbb{Z}$. Function $\varphi^{[n]}$ is compactly supported so there exists $r \in \mathbb{R}_+$ so that $\text{supp } \varphi^{[n]} \subset \overline{B}_{\mathbb{R}^n}(0; r)$.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Now

$$|g(\mathbf{x}) - g(\mathbf{y})| = \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} a_{j,\mathbf{k}} \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) - \sum_{\ell \in \mathbb{Z}^n} a_{j,\ell} \varphi^{[n]}(2^j \mathbf{y} - \ell) \right| \quad (37a)$$

$$= \left| \sum_{\mathbf{k} \in \mathbb{Z}^n} a_{j,\mathbf{k}} \left(\varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) - \varphi^{[n]}(2^j \mathbf{y} - \mathbf{k}) \right) \right| \quad (37b)$$

for all $j \in \mathbb{Z}$. Since f is compactly supported all the series in Equations (37a) and (37b) contain only finite number of nonzero terms. Therefore

$$\begin{aligned} |g(\mathbf{x}) - g(\mathbf{y})| &\leq \|\mathbf{a}_j\|_\infty \sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) - \varphi^{[n]}(2^j \mathbf{y} - \mathbf{k}) \right| \\ &\leq \|g\|_\infty \sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) - \varphi^{[n]}(2^j \mathbf{y} - \mathbf{k}) \right| \end{aligned}$$

for all $j \in \mathbb{Z}$ where the series contain only finite number of nonzero terms. Function $\varphi^{[n]}$ is uniformly continuous and hence $|\varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) - \varphi^{[n]}(2^j \mathbf{y} - \mathbf{k})| \rightarrow 0$ as $j \rightarrow -\infty$ for each $\mathbf{k} \in \mathbb{Z}^n$.

Let $m = \max\{\|\mathbf{x}\|, \|\mathbf{y}\|\}$. Suppose that $j \in \mathbb{Z}$, $j \leq 0$, $\mathbf{m} \in \mathbb{Z}^n$, and $\|\mathbf{m}\|_2 > r + m$. Now $2^j \|\mathbf{x}\| \leq 2^j m \leq m$ and $\|2^j \mathbf{x} - \mathbf{m}\| \geq |2^j \|\mathbf{x}\| - \|\mathbf{m}\|_2| = \|\mathbf{m}\|_2 - 2^j \|\mathbf{x}\| > r + m - m = r$. Hence $\varphi^{[n]}(2^j \mathbf{x} - \mathbf{m}) = 0$. We also get $\varphi^{[n]}(2^j \mathbf{y} - \mathbf{m}) = 0$ similarly. Let

$$\mathbf{d}[\mathbf{k}] := \begin{cases} 2\|\varphi^{[n]}\|_\infty; & \text{if } \|\mathbf{k}\|_2 \leq r + m \\ 0; & \text{otherwise} \end{cases}$$

for all $\mathbf{k} \in \mathbb{Z}^n$. Now $|\varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) - \varphi^{[n]}(2^j \mathbf{y} - \mathbf{k})| \leq \mathbf{d}[\mathbf{k}]$ for all $\mathbf{k} \in \mathbb{Z}^n$, $j \in \mathbb{Z}$, $j \leq 0$ and

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] < \infty.$$

Thus by the Dominated Convergence Theorem for Series [28]

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \left| \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) - \varphi^{[n]}(2^j \mathbf{y} - \mathbf{k}) \right| \rightarrow 0 \text{ as } j \rightarrow -\infty.$$

Hence $|g(\mathbf{x}) - g(\mathbf{y})| = 0$. Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ were arbitrary so g is a constant function. Consequently

$$\bigcap_{j \in \mathbb{Z}} V_{n,j}^{(u)} = K.$$

□

Definition 6.5. When $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0, 1\}^n$ define

$$\begin{aligned} W_{n,\mathbf{s},j}^{(u)} &:= \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}(\mathbf{x}) : \mathbf{a} \in l^\infty(\mathbb{Z}^n, K) \right\} \\ \|f|W_{n,\mathbf{s},j}^{(u)}\| &:= \|f\|_\infty, \quad f \in W_{n,\mathbf{s},j}^{(u)}. \end{aligned}$$

When $j \in \mathbb{Z}$ we have $V_{n,j}^{(u)} =_{\text{n.s.}} W_{n,0_n,j}^{(u)}$. By Lemma 3.10 function $\eta_{n,s,j}^{(u)} : l^\infty(\mathbb{Z}^n, K) \rightarrow W_{n,s,j}^{(u)}$ defined by

$$\eta_{n,s,j}^{(u)}(\mathbf{a}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \psi_{s,j,\mathbf{k}}^{[n]}$$

for all $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$ is a topological isomorphism from $l^\infty(\mathbb{Z}^n, K)$ onto $W_{n,s,j}^{(u)}$ and

$$\|\mathbf{a}\|_\infty \leq \left\| \eta_{n,s,j}^{(u)}(\mathbf{a}) \right\|_\infty \leq N_{\text{cover}} \left(\psi_{\mathbf{s}}^{[n]} \right) \left\| \psi_{\mathbf{s}}^{[n]} \right\|_\infty \|\mathbf{a}\|_\infty \quad (38)$$

for all $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$, $\mathbf{s} \in \{0, 1\}^n$, and $j \in \mathbb{Z}$.

Definition 6.6. When $\mathbf{s} \in \{0, 1\}^n$ and $j \in \mathbb{Z}$ define

$$\left(Q_{n,s,j}^{(u)} f \right) (\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \left\langle \tilde{\psi}_{s,j,\mathbf{k}}^{[n]}, f \right\rangle \psi_{s,j,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $f \in C_u(\mathbb{R}^n, K)$. When $j \in \mathbb{Z}$ define $P_{n,j}^{(u)} := Q_{n,0_n,j}^{(u)}$.

Operator $Q_{n,s,j}^{(u)}$ is a continuous projection of $C_u(\mathbb{R}^n, K)$ onto $W_{n,s,j}^{(u)}$ for each $\mathbf{s} \in \{0, 1\}^n$ and $j \in \mathbb{Z}$. When $\mathbf{s} \in \{0, 1\}^n$ operators $Q_{n,s,j}^{(u)}$, $j \in \mathbb{Z}$, are uniformly bounded by

$$\left\| Q_{n,s,j}^{(u)} \right\| \leq \left\| \tilde{\psi}_{\mathbf{s}}^{[n]} \right\| \left\| \psi_{\mathbf{s}}^{[n]} \right\|_\infty N_{\text{cover}}(\psi_{\mathbf{s}}^{[n]})$$

for all $j \in \mathbb{Z}$. Operators $P_{n,j}^{(u)}$, $j \in \mathbb{Z}$, are uniformly bounded by

$$\left\| P_{n,j}^{(u)} \right\| \leq \left\| \varphi^{[n]} \right\|_\infty N_{\text{cover}}(\varphi^{[n]}) \quad (39)$$

for all $j \in \mathbb{Z}$.

Lemma 6.7. Let $j \in \mathbb{Z}$, $\mathbf{s} \in \{0, 1\}^n$, $\mathbf{k} \in \mathbb{Z}^n$, and $f \in C_u(\mathbb{R}^n)$. Then $\langle \tilde{\psi}_{s,j,\mathbf{k}}^{[n]}, f \rangle = \langle \tilde{\psi}_{s,j,\mathbf{k}}^{[n]}, P_{n,j+1}^{(u)} f \rangle$.

Proof. Use Equation (28). □

Lemma 6.8. Let $j, j' \in \mathbb{Z}$, $j < j'$, and $\mathbf{s} \in \{0, 1\}^n$. Then $Q_{n,s,j}^{(u)} = Q_{n,s,j}^{(u)} \circ P_{n,j'}^{(u)}$.

Proof. Use Lemma 6.7. □

Lemma 6.9. Let $j \in \mathbb{Z}$. Let $\mathbf{s}, \mathbf{t} \in \{0, 1\}^n$ and $\mathbf{s} \neq \mathbf{t}$. Then $W_{n,s,j}^{(u)} \cap W_{n,t,j}^{(u)} = \{0\}$.

Lemma 6.10. Let $j \in \mathbb{Z}$, $\mathbf{s} \in \{0, 1\}^n$, and $\ell \in \mathbb{Z}^n$. Let $\mathbf{a} \in l^\infty(\mathbb{Z}^n)$ and

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi_{j+1,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$. Then

$$\left(Q_{n,s,j}^{(u)} f \right) \left(\frac{\ell}{2^{j+1}} \right) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \sum_{\mathbf{z} \in \mathbb{Z}^n} \tilde{g}_{s,\mathbf{k}-2\mathbf{z}}^{[n]} g_{s,\ell-2\mathbf{z}}^{[n]}.$$

Proof. Use Equations (31) and (32). □

Definition 6.11. When $j \in \mathbb{Z}$ define

$$W_{n,j}^{(u)} :=_{\text{n.s.}} \sum_{\mathbf{s} \in J_+(n)} W_{n,\mathbf{s},j}^{(u)}$$

and

$$Q_{n,j}^{(u)} := \sum_{\mathbf{s} \in J_+(n)} Q_{n,\mathbf{s},j}^{(u)}.$$

Lemma 6.12. Let $j \in \mathbb{Z}$. Then

- (i) $\forall \mathbf{s} \in \{0,1\}^n, \mathbf{t} \in \{0,1\}^n : \mathbf{s} \neq \mathbf{t} \implies \forall f \in W_{n,\mathbf{s},j}^{(u)} : Q_{n,\mathbf{t},j}^{(u)} f = 0$
- (ii) $\forall j' \in \mathbb{Z}, f \in V_{n,j'}^{(u)} : j' \leq j \implies Q_{n,j}^{(u)} f = 0$
- (iii) $\forall j' \in \mathbb{Z}, f \in W_{n,j'}^{(u)} : j' \neq j \implies Q_{n,j}^{(u)} f = 0$
- (iv) $\forall j' \in \mathbb{Z}, f \in W_{n,j'}^{(u)} : j' \geq j \implies P_{n,j}^{(u)} f = 0.$

Proof.

- (i) Let $\mathbf{s}, \mathbf{t} \in \{0,1\}^n$ and $\mathbf{s} \neq \mathbf{t}$. Let $f \in W_{n,\mathbf{s},j}^{(u)}$. Now

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{a} \in l^\infty(\mathbb{Z}^n)$. By Lemma 3.21 we have

$$\langle \tilde{\psi}_{\mathbf{t},j,\ell}^{[n]}, f \rangle = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \langle \tilde{\psi}_{\mathbf{t},j,\ell}^{[n]}, \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \rangle = 0$$

for all $\ell \in \mathbb{Z}^n$. Hence $Q_{n,\mathbf{t},j}^{(u)} f = 0$.

- (ii) If $j' < j$ then $V_{n,j'}^{(u)} \subset V_{n,j}^{(u)} = W_{n,\mathbf{0}_n,j}^{(u)}$ and proposition (ii) follows from proposition (i).
- (iii) If $j < j'$ and $g \in W_{n,j'}^{(u)}$, then $Q_{n,j}^{(u)} g = Q_{n,j}^{(u)} P_{n,j'}^{(u)} g$. Since $P_{n,j'}^{(u)} = Q_{n,\mathbf{0}_n,j'}^{(u)}$ it follows that $Q_{n,j}^{(u)} g = 0$ by proposition (i). If $j > j'$ and $g \in W_{n,j'}^{(u)}$, then $W_{n,j'}^{(u)} \subset V_{n,j}^{(u)}$ and $Q_{n,j}^{(u)} g = 0$ by proposition (ii).
- (iv) Let $j' \in \mathbb{Z}, j' \geq j$. Suppose that $w \in W_{n,j'}^{(u)}$. Since $P_{n,j'}^{(u)} = Q_{n,\mathbf{0}_n,j'}^{(u)}$ it follows that $P_{n,j}^{(u)} w = P_{n,j}^{(u)} P_{n,j'}^{(u)} w = 0$ by proposition (i).

□

Lemma 6.13. Let $j \in \mathbb{Z}$. Then

$$V_{n,j+1}^{(u)} =_{\text{n.s.}} V_{n,j}^{(u)} \dot{+} W_{n,j}^{(u)} =_{\text{n.s.}} V_{n,j}^{(u)} \dot{+} \sum_{\mathbf{s} \in J_+(n)} W_{n,\mathbf{s},j}^{(u)} =_{\text{n.s.}} \sum_{\mathbf{s} \in \{0,1\}^n} W_{n,\mathbf{s},j}^{(u)}.$$

Proof. By Lemma 6.9 the sums in the lemma are direct.

Suppose that $f \in V_{n,j+1}^{(u)}$. Now

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi_{j+1,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ where $\mathbf{a} \in l^\infty(\mathbb{Z}^n, K)$. Let $f_{\mathbf{s}} := Q_{n,\mathbf{s},j}^{(\mathbf{u})} f \in W_{n,\mathbf{s},j}^{(\mathbf{u})}$ for each $\mathbf{s} \in \{0,1\}^n$. Suppose that $\ell \in \mathbb{Z}^n$. By Lemma 6.10

$$\begin{aligned} \sum_{\mathbf{s} \in \{0,1\}^n} f_{\mathbf{s}} \left(\frac{\ell}{2^{j+1}} \right) &= \sum_{\mathbf{s} \in \{0,1\}^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \sum_{\mathbf{m} \in \mathbb{Z}^n} \tilde{g}_{\mathbf{s},\mathbf{k}-2\mathbf{m}}^{[n]} g_{\mathbf{s},\ell-2\mathbf{m}}^{[n]} \\ &= \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \sum_{\mathbf{s} \in \{0,1\}^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} \tilde{g}_{\mathbf{s},\mathbf{k}-2\mathbf{m}}^{[n]} g_{\mathbf{s},\ell-2\mathbf{m}}^{[n]}. \end{aligned}$$

By Lemma 5.15

$$\sum_{\mathbf{s} \in \{0,1\}^n} f_{\mathbf{s}} \left(\frac{\ell}{2^{j+1}} \right) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \delta_{\mathbf{k},\ell} = \mathbf{a}[\ell]$$

Consequently

$$\left(\sum_{\mathbf{s} \in \{0,1\}^n} f_{\mathbf{s}} \right) (\mathbf{x}) = \sum_{\ell \in \mathbb{Z}^n} \left(\sum_{\mathbf{s} \in \{0,1\}^n} f_{\mathbf{s}} \right) \left(\frac{\ell}{2^{j+1}} \right) \varphi_{j+1,\ell}^{[n]}(\mathbf{x}) = \sum_{\ell \in \mathbb{Z}^n} \mathbf{a}[\ell] \varphi_{j+1,\ell}^{[n]}(\mathbf{x}).$$

Hence

$$f = \sum_{\mathbf{s} \in \{0,1\}^n} f_{\mathbf{s}}.$$

□

Lemma 6.14.

- (i) $Q_{n,j}^{(\mathbf{u})}$ is a continuous projection of $C_u(\mathbb{R}^n, K)$ onto $W_{n,j}^{(\mathbf{u})}$ for each $j \in \mathbb{Z}$.
- (ii) We have $Q_{n,j}^{(\mathbf{u})} = P_{n,j+1}^{(\mathbf{u})} - P_{n,j}^{(\mathbf{u})}$ for all $j \in \mathbb{Z}$.
- (iii) Operators $Q_{n,j}^{(\mathbf{u})}$, $j \in \mathbb{Z}$, are uniformly bounded by

$$\left\| Q_{n,j}^{(\mathbf{u})} \right\| \leq 2N_{\text{cover}}(\varphi^{[n]}) \left\| \varphi^{[n]} \right\|_{\infty}.$$

Proof. Use Lemmas 6.8, 6.12, and 6.13 and Equation (39). □

As $(\tilde{\varphi}^{[n]}, \varphi^{[n]})$ spans all the polynomials of degree 0 we have

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \varphi^{[n]}(\mathbf{x} - \mathbf{k}) = 1 \tag{40}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Theorem 6.15. *There exists $c_1 \in \mathbb{R}_+$ and $c_2 \in \mathbb{R}_+$ so that*

$$\left\| f - P_{n,j}^{(\mathbf{u})} f \right\|_{\infty} \leq c_1 \omega(f; 2^{-j} c_2)$$

for all $f \in C_u(\mathbb{R}^n)$ and $j \in \mathbb{Z}$.

Proof. Let $f \in C_u(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. Define $r_1 := r_{\text{supp}}(\varphi^{[n]})$. Let $\mathbf{x} \in \mathbb{R}^n$. Define $I := I_{\text{trans}}(\varphi^{[n]}, 2^j \mathbf{x})$. Now by Equation (40) we have

$$\begin{aligned} (f - P_{n,j}^{(u)} f)(\mathbf{x}) &= \sum_{\mathbf{k} \in \mathbb{Z}^n} \left(f(\mathbf{x}) - f\left(\frac{\mathbf{k}}{2^j}\right) \right) \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) \\ &= \sum_{\mathbf{k} \in I} \left(f(\mathbf{x}) - f\left(\frac{\mathbf{k}}{2^j}\right) \right) \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k}) \end{aligned}$$

Let $\ell \in I$. Now $2^j \mathbf{x} - \ell \in \overline{B}_{\mathbb{R}^n}(0; r_1)$ from which it follows that $\mathbf{x} - 2^{-j} \ell \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j} r_1)$. Hence

$$\left| f(\mathbf{x}) - f\left(\frac{\ell}{2^j}\right) \right| \leq \omega(f; 2^{-j} r_1). \quad (41)$$

Let $c_1 := N_{\text{cover}}(\varphi^{[n]}) \|\varphi^{[n]}\|_\infty$ and $c_2 := r_1 = r_{\text{supp}}(\varphi^{[n]})$. By Equation (41) we have

$$\left| (f - P_{n,j}^{(u)} f)(\mathbf{x}) \right| \leq c_1 \omega(f; 2^{-j} c_2).$$

□

Theorem 6.16. *Let $f \in C_u(\mathbb{R}^n)$. Now*

$$\lim_{j \rightarrow \infty} \|f - P_{n,j}^{(u)} f\|_\infty = 0.$$

Proof. See also [7, theorem 3.2] and [16, theorem 2.4]. Let $j \in \mathbb{Z}$. By Theorem 6.15 we have $\|f - P_{n,j}^{(u)} f\|_\infty \leq c_1 \omega(f; 2^{-j} c_2)$ where c_1 and c_2 do not depend on j or f . Since f is uniformly continuous

$$\lim_{t \rightarrow 0} \omega(f; t) = 0$$

and hence

$$\lim_{j \rightarrow \infty} \|f - P_{n,j}^{(u)} f\|_\infty = 0.$$

□

Theorem 6.17.

$$\overline{\bigcup_{j \in \mathbb{Z}} V_{n,j}^{(u)}} = C_u(\mathbb{R}^n, K).$$

Proof. Since $V_{n,j}^{(u)} \in C_u(\mathbb{R}^n, K)$ for all $j \in \mathbb{Z}$ it follows that $\overline{\bigcup_{j \in \mathbb{Z}} V_{n,j}^{(u)}} \subset C_u(\mathbb{R}^n, K)$. Let $f \in C_u(\mathbb{R}^n, K)$. Now $(P_{n,j}^{(u)} f)_{j=0}^\infty$ is a sequence in the set $\bigcup_{j \in \mathbb{Z}} V_{n,j}^{(u)}$. It follows from Theorem 6.16 that $P_{n,j}^{(u)} f \rightarrow f$ as $j \rightarrow \infty$. Consequently $f \in \overline{\bigcup_{j \in \mathbb{Z}} V_{n,j}^{(u)}}$. □

By Theorem 6.16 we have

$$f = P_{n,j_0}^{(u)} f + \sum_{j=j_0}^\infty Q_{n,j}^{(u)} f \quad (42)$$

for all $f \in C_u(\mathbb{R}^n, K)$. If

$$f = v + \sum_{j=j_0}^\infty w_j$$

where $v \in V_{n,j_0}^{(u)}$ and $w_j \in W_{n,j}^{(u)}$ for all $j \in \mathbb{Z}$, $j \geq j_0$, it follows from Lemmas 6.14 (ii) and 6.12 that $v = P_{n,j_0}^{(u)} f$ and $w_j = Q_{n,j}^{(u)} f$ for all $j \in \mathbb{Z}$, $j \geq j_0$.

Definition 6.18. When $j \in \mathbb{Z}$ define

$$\eta_{n,j}^{(u)}(\mathbf{a}) := \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{s} \in J_+(n)} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{s}, \mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $\mathbf{a} \in l^\infty(J_+(n) \times \mathbb{Z}^n, K)$.

Lemma 6.19.

- (i) $W_{n,j}^{(u)} \subset_{c.s.} C_u(\mathbb{R}^n, K)$ for all $j \in \mathbb{Z}$.
- (ii) Function $\eta_{n,j}^{(u)}$ is a topological isomorphism from $l^\infty(J_+(n) \times \mathbb{Z}^n, K)$ onto $W_{n,j}^{(u)}$.
- (iii) There exists $c_1 \in \mathbb{R}_+$ so that $\|\eta_{n,j}^{(u)}\| \leq c_1$ for all $j \in \mathbb{Z}$.
- (iv) There exists $c_2 \in \mathbb{R}_+$ so that $\|(\eta_{n,j}^{(u)})^{-1}\| \leq c_2$ for all $j \in \mathbb{Z}$.

Proof. Use Lemma 3.20 and Definitions 6.5 and 6.11. □

Definition 6.20. Let $n \in \mathbb{N} + 2$. When $j \in \mathbb{Z}$ and $l \in \mathbb{N}$ define

$$v_{j,l}(\mathbf{x}, \mathbf{h}) := \sum_{p=0}^l \left(\varphi^{[n]}(2^j \mathbf{x} - \sigma_c^{[n]}(p) - \lfloor 2^j \mathbf{x} \rfloor + 2^j \mathbf{h}) - \varphi^{[n]}(2^j \mathbf{x} - \sigma_c^{[n]}(p) - \lfloor 2^j \mathbf{x} \rfloor) \right)$$

for all \mathbf{x}, \mathbf{h} in \mathbb{R}^n .

Lemma 6.21. Let $n \in \mathbb{N} + 2$ and $t \in \mathbb{R}_+$. Then there exists $c_1 \in \mathbb{R}_+$, which may depend on t , so that

$$\sum_{l=0}^{\infty} |v_{j,l}(\mathbf{x}, \mathbf{h})| \leq c_1$$

for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}t)$, and $j \in \mathbb{Z}$.

Proof. Let $\mathbf{x}_1 \in \mathbb{R}^n$. Let

$$I := \{\mathbf{k} \in \mathbb{Z}^n : \exists \mathbf{y} \in \overline{B}_{\mathbb{R}^n}(2^j \mathbf{x}_1; t) : \varphi^{[n]}(\mathbf{y} - \mathbf{k}) \neq 0\}.$$

There exists $r_1 \in \mathbb{R}_+$ so that $\text{supp } \varphi^{[n]} \subset \overline{B}_{\mathbb{R}^n}(0; r_1)$. Let $\mathbf{k}_1 \in I$. Now $\varphi^{[n]}(\mathbf{y} - \mathbf{k}_1) \neq 0$ for some $\mathbf{y} \in \overline{B}_{\mathbb{R}^n}(2^j \mathbf{x}_1; t)$. Furthermore,

$$\|\mathbf{k}_1 - \lfloor 2^j \mathbf{x}_1 \rfloor\| \leq \|\mathbf{k}_1 - \mathbf{y}\| + \|\mathbf{y} - 2^j \mathbf{x}_1\| + \|2^j \mathbf{x}_1 - \lfloor 2^j \mathbf{x}_1 \rfloor\| \leq r_2 := r_1 + t + \sqrt{n}.$$

Now $I \subset K := I_{\text{rect}}(\lfloor 2^j \mathbf{x}_1 \rfloor - \lceil r_2 \rceil, \lfloor 2^j \mathbf{x}_1 \rfloor + \lceil r_2 \rceil)$ and $m := \#K = (2\lceil r_2 \rceil + 1)^n$. As $\sigma_c^{[n]}$ increases along zero-centred cubes we have $\sigma_c^{[n]}[Z_0(\#K - 1)] + \lfloor 2^j \mathbf{x}_1 \rfloor = K$. If $l_1 \in \mathbb{N} + m$ we have

$$v_{j,l_1}(\mathbf{x}_1, \mathbf{h}) = \sum_{p=0}^{l_1} \left(\varphi^{[n]}(2^j \mathbf{x}_1 + 2^j \mathbf{h} - \sigma_c^{[n]}(p) - \lfloor 2^j \mathbf{x}_1 \rfloor) - \varphi^{[n]}(2^j \mathbf{x}_1 - \sigma_c^{[n]}(p) - \lfloor 2^j \mathbf{x}_1 \rfloor) \right) = 1 - 1 = 0$$

for all $\mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}t)$. Hence

$$\begin{aligned} \sum_{l=0}^{\infty} |v_{j,l}(\mathbf{x}_1, \mathbf{h})| &= \sum_{l=0}^{m-1} |v_{j,l}(\mathbf{x}_1, \mathbf{h})| \\ &\leq \sum_{l=0}^{m-1} \sum_{p=0}^l \left| \varphi^{[n]}(2^j \mathbf{x}_1 + 2^j \mathbf{h} - \sigma_c^{[n]}(p) - \lfloor 2^j \mathbf{x}_1 \rfloor) - \varphi^{[n]}(2^j \mathbf{x}_1 - \sigma_c^{[n]}(p) - \lfloor 2^j \mathbf{x}_1 \rfloor) \right| \\ &\leq c_1 := 2m^2 \|\varphi^{[n]}\|_{\infty} \end{aligned}$$

for all $\mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}t)$. □

Theorem 6.22. *Let $n \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$. There exists a constant $c_1 \in \mathbb{R}_+$, which may depend on t , so that*

$$\forall f \in C_u(\mathbb{R}^n), j \in \mathbb{Z} : \omega(P_{n,j}^{(u)} f; 2^{-j}t) \leq c_1 \omega(f; 2^{-j}\sqrt{n}).$$

Proof. The proof of case $n = 1$ is similar to the proof in [16, section 7.1]. We will assume that $n \geq 2$ in the sequel.

Let $f \in C_u(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. Let $\mathbf{a} := (f(2^{-j}\mathbf{k}))_{\mathbf{k} \in \mathbb{Z}^n}$. Now

$$\left(P_{n,j}^{(u)} f\right)(\mathbf{x} + \mathbf{h}) - \left(P_{n,j}^{(u)} f\right)(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \left(\varphi^{[n]}(2^j \mathbf{x} + 2^j \mathbf{h} - \mathbf{k}) - \varphi^{[n]}(2^j \mathbf{x} - \mathbf{k})\right)$$

for all $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$. Assume that $\mathbf{x}_1 \in \mathbb{R}^n$ and $\mathbf{h}_1 \in \overline{B}_{\mathbb{R}^n}(0; 2^{-j}t)$. Let $\eta(p) := \sigma_c^{[n]}(p) + \lfloor 2^j \mathbf{x}_1 \rfloor$ for all $p \in \mathbb{N}$. Define $v_l := v_{j,l}(\mathbf{x}_1, \mathbf{h}_1)$ and

$$w_l := \mathbf{a}[\sigma_c^{[n]}(l)] \left(\varphi^{[n]}(2^j \mathbf{x}_1 + 2^j \mathbf{h}_1 - \eta(l)) - \varphi^{[n]}(2^j \mathbf{x}_1 - \eta(l))\right)$$

for all $l \in \mathbb{N}$. Now

$$\begin{aligned} w_l &= \mathbf{a}[\sigma_c^{[n]}(l)] \left(\sum_{p=0}^l \left(\varphi^{[n]}(2^j \mathbf{x}_1 + 2^j \mathbf{h}_1 - \eta(p)) - \varphi^{[n]}(2^j \mathbf{x}_1 - \eta(p)) \right) \right. \\ &\quad \left. - \sum_{p=0}^{l-1} \left(\varphi^{[n]}(2^j \mathbf{x}_1 + 2^j \mathbf{h}_1 - \eta(p)) - \varphi^{[n]}(2^j \mathbf{x}_1 - \eta(p)) \right) \right) \\ &= \mathbf{a}[\sigma_c^{[n]}(l)] v_l - \mathbf{a}[\sigma_c^{[n]}(l)] v_{l-1} \end{aligned}$$

for all $l \in \mathbb{Z}_+$. Furthermore,

$$\begin{aligned} \sum_{l=1}^{\infty} w_l &= \sum_{l=1}^{\infty} \mathbf{a}[\sigma_c^{[n]}(l)] v_l - \sum_{l=1}^{\infty} \mathbf{a}[\sigma_c^{[n]}(l)] v_{l-1} = \sum_{l=1}^{\infty} \mathbf{a}[\sigma_c^{[n]}(l)] v_l - \sum_{l=0}^{\infty} \mathbf{a}[\sigma_c^{[n]}(l+1)] v_l \\ &= -\mathbf{a}[\sigma_c^{[n]}(1)] v_0 + \sum_{l=1}^{\infty} \left(\mathbf{a}[\sigma_c^{[n]}(l)] - \mathbf{a}[\sigma_c^{[n]}(l+1)] \right) v_l. \end{aligned}$$

We also have $w_0 = \mathbf{a}[\sigma_c^{[n]}(0)] v_0$ from which it follows that

$$\begin{aligned} z &:= \sum_{l=0}^{\infty} w_l = \mathbf{a}[\sigma_c^{[n]}(0)] v_0 - \mathbf{a}[\sigma_c^{[n]}(1)] v_0 + \sum_{l=1}^{\infty} \left(\mathbf{a}[\sigma_c^{[n]}(l)] - \mathbf{a}[\sigma_c^{[n]}(l+1)] \right) v_l \\ &= \sum_{l=0}^{\infty} \left(\mathbf{a}[\sigma_c^{[n]}(l)] - \mathbf{a}[\sigma_c^{[n]}(l+1)] \right) v_l. \end{aligned}$$

Consequently

$$|z| \leq \left(\sup_{l \in \mathbb{N}} \left| \mathbf{a}[\sigma_c^{[n]}(l)] - \mathbf{a}[\sigma_c^{[n]}(l+1)] \right| \right) \cdot \sum_{l=0}^{\infty} |v_l|$$

Function $\sigma_c^{[n]}$ preserves neighbours from which it follows that

$$\begin{aligned} & \left\| \sigma_c^{[n]}(l) - \sigma_c^{[n]}(l+1) \right\|_\infty = 1 \\ \implies & \left\| \sigma_c^{[n]}(l) - \sigma_c^{[n]}(l+1) \right\|_2 \leq \sqrt{n} \\ \implies & \left\| 2^{-j} \sigma_c^{[n]}(l) - 2^{-j} \sigma_c^{[n]}(l+1) \right\|_2 \leq 2^{-j} \sqrt{n}. \end{aligned}$$

for all $l \in \mathbb{N}$. Consequently

$$\sup_{l \in \mathbb{N}} \left| \mathbf{a} \left[\sigma_c^{[n]}(l) \right] - \mathbf{a} \left[\sigma_c^{[n]}(l+1) \right] \right| \leq \omega(f; 2^{-j} \sqrt{n}). \quad (43)$$

It follows from Lemma 6.21 and Equation (43) that

$$\left| \left(P_{n,j}^{(u)} f \right) (\mathbf{x}_1 + \mathbf{h}_1) - \left(P_{n,j}^{(u)} f \right) (\mathbf{x}_1) \right| = |z| \leq c_1 \omega(f; 2^{-j} \sqrt{n})$$

□

7 Compactly Supported Interpolating MRA of $C_0(\mathbb{R}^n, K)$

We shall assume that $K = \mathbb{R}$ or $K = \mathbb{C}$ throughout this section. We shall also assume that $\varphi \in C_{\text{com}}(\mathbb{R}, K)$ is a function for which conditions (MSF.1) and (MSF.2) hold and $(\tilde{\varphi}, \varphi)$ spans all polynomials of degree 0. Unless otherwise stated, we shall assume that the same values of K and φ are used throughout this section.

7.1 General

Donoho [16] constructs projection operators and gives convergence results for interpolating wavelets on $C_0(\mathbb{R})$.

Definition 7.1. Let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$. Let $\varphi \in C_{\text{com}}(\mathbb{R}, K)$ be a function for which conditions (MSF.1) and (MSF.2) hold and let $\varphi^{[n]}$ be defined by Definition 5.10. Define

$$\begin{aligned} V_{n,j}^{(0)} &:= \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi^{[n]}(2^j \cdot -\mathbf{k}) : \mathbf{a} \in c_0(\mathbb{Z}^n, K) \right\} \\ \|f|V_{n,j}^{(0)}\| &:= \|f\|_\infty, \quad f \in V_{n,j}^{(0)} \end{aligned} \quad (44)$$

for each $j \in \mathbb{Z}$.

The infinite sum in Equation (44) converges unconditionally by Lemma 3.7.

Definition 7.2. Assume that $\varphi^{[n]}$ is an n -dimensional tensor product mother scaling function. Let spaces $V_{n,j}^{(0)}$, $j \in \mathbb{Z}$, be defined by Definition 7.1. We call $\{V_{n,j}^{(0)} : j \in \mathbb{Z}\}$ an interpolating tensor product MRA of $C_0(\mathbb{R}^n, K)$ generated by $\varphi^{[n]}$ provided that the following conditions are satisfied:

$$(MRA2.1) \quad \forall j \in \mathbb{Z} : V_{n,j}^{(0)} \subset V_{n,j+1}^{(0)}$$

$$(MRA2.2) \quad \overline{\bigcup_{j \in \mathbb{Z}} V_{n,j}^{(0)}} = C_0(\mathbb{R}^n, K)$$

$$(MRA2.3) \quad \bigcap_{j \in \mathbb{Z}} V_{n,j}^{(0)} = \{0\}$$

$$(MRA2.4) \quad \forall j \in \mathbb{Z}, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(0)} \iff f(2 \cdot) \in V_{n,j+1}^{(0)}$$

$$(MRA2.5) \quad \forall j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n, f \in K^{\mathbb{R}^n} : f \in V_{n,j}^{(0)} \iff f(\cdot - 2^{-j}\mathbf{k}) \in V_{n,j}^{(0)}$$

$$(MRA2.6) \quad \forall \mathbf{k} \in \mathbb{Z}^n : \varphi^{[n]}(\mathbf{k}) = \delta_{\mathbf{k},0}$$

Note that in this definition the intersection of spaces $V_{n,j}^{(0)}$ is $\{0\}$ instead of \mathbb{C} (all complex valued functions on \mathbb{R}^n) as in [7] since here the MRA is constructed for functions vanishing at infinity. Donoho [16] does not require the mother scaling function φ to be compactly supported but φ has to be of rapid decay in his construction. He also includes requirements for the regularity and polynomial span of φ , which are needed for the norm equivalences to Besov and Triebel-Lizorkin spaces, into the definition of the MRA.

Lemma 7.3. *Under the condition that $\varphi^{[n]}$ is an n -dimensional tensor product mother scaling function the conditions (MRA2.1), (MRA2.4), (MRA2.5), and (MRA2.6) are true.*

See also Lemma 6.3.

7.2 Basic Definitions for the Univariate MRA of $C_0(\mathbb{R}, K)$

By Lemma 3.9 spaces $V_{1,j}^{(0)}$, $j \in \mathbb{Z}$, are closed subspaces of $C_0(\mathbb{R}, K)$.

Definition 7.4. When $j \in \mathbb{Z}$ define operator $P_{1,j}^{(0)} : C_0(\mathbb{R}, K) \rightarrow V_{1,j}^{(0)}$ by

$$P_{1,j}^{(0)} f := \sum_{k \in \mathbb{Z}} \langle \tilde{\varphi}_{j,k}, f \rangle \varphi_{j,k}. \quad (45)$$

The infinite sum in (45) converges unconditionally by Lemma 3.7. By Definition 7.1

$$\sum_{k \in \mathbb{Z}} f\left(\frac{k}{2^j}\right) \varphi(2^j \cdot - k) \in V_{1,j}^{(0)}$$

for all $f \in C_0(\mathbb{R}, K)$ and $j \in \mathbb{Z}$. Hence functions $P_{1,j}^{(0)}$, $j \in \mathbb{Z}$, are well defined. Functions $P_{1,j}^{(0)}$, $j \in \mathbb{Z}$, are linear by their definition. We have

$$\left\| \left(f\left(\frac{k}{2^j}\right) \right)_{k \in \mathbb{Z}} \right\|_{\infty} \leq \|f\|_{\infty}$$

for all $f \in C_0(\mathbb{R}, K)$. Hence by Lemma 3.7 functions $P_{1,j}^{(0)}$, $j \in \mathbb{Z}$, are continuous and uniformly bounded by

$$\left\| P_{1,j}^{(0)} \right\| \leq N_{\text{cover}}(\varphi) \|\varphi\|_{\infty} \quad (46)$$

for all $j \in \mathbb{Z}$. It follows from Definitions 7.1 and 7.4 that operator $P_{1,j}^{(0)}$ is a projection onto Banach space $V_{1,j}^{(0)}$ for each $j \in \mathbb{Z}$.

Lemma 7.5. $P_{1,j}^{(0)} = P_{1,j}^{(0)} \circ P_{1,j'}^{(0)}$ for all $j, j' \in \mathbb{Z}$, $j' \geq j$.

Proof. Let $j, j' \in \mathbb{Z}$ and $j' \geq j$. Let $f \in C_0(\mathbb{R}, K)$. Now

$$\begin{aligned} P_{1,j}^{(0)}(P_{1,j'}^{(0)}f) &= \sum_{k \in \mathbb{Z}} \left\langle \tilde{\varphi}_{j,k}, \sum_{k' \in \mathbb{Z}} \langle \tilde{\varphi}_{j',k'}, f \rangle \varphi_{j',k'} \right\rangle \varphi_{j,k} = \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \langle \tilde{\varphi}_{j',k'}, f \rangle \langle \tilde{\varphi}_{j,k}, \varphi_{j',k'} \rangle \varphi_{j,k} \\ &= \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} \langle \tilde{\varphi}_{j',k'}, f \rangle \delta_{k', 2j' - jk} \varphi_{j,k} = \sum_{k \in \mathbb{Z}} \langle \tilde{\varphi}_{j', 2j' - jk}, f \rangle \varphi_{j,k} = \sum_{k \in \mathbb{Z}} \langle \tilde{\varphi}_{j,k}, f \rangle \varphi_{j,k} \\ &= P_{1,j}^{(0)}f. \end{aligned}$$

□

We get the following theorem as a direct consequence of Theorem 6.16.

Theorem 7.6. *Let $f \in C_0(\mathbb{R})$. Now*

$$\lim_{j \rightarrow \infty} \|f - P_{1,j}^{(0)}f\|_{\infty} = 0.$$

Definition 7.7. Define $W_{1,j}^{(0)} = \{f - P_{1,j}^{(0)}f : f \in V_{j+1}\}$ for all $j \in \mathbb{Z}$.

Lemma 7.8. $W_{1,j}^{(0)}$ is a closed subspace of $V_{1,j+1}^{(0)}$ for each $j \in \mathbb{Z}$ and $V_{1,j}^{(0)} \cap W_{1,j}^{(0)} = \{0\}$ for all $j \in \mathbb{Z}$.

Proof. If $f \in V_{1,j}^{(0)}$ then $P_{1,j}^{(0)}f = f$. If $g \in W_{1,j}^{(0)}$ and $g \neq 0$ then $g = h - P_{1,j}^{(0)}h$ where $h \in V_{j+1}$ and $P_{1,j}^{(0)}g = 0$. So $P_{1,j}^{(0)}g \neq g$ and $g \notin V_{1,j}^{(0)}$. Hence $V_{1,j}^{(0)} \cap W_{1,j}^{(0)} = \{0\}$.

Let $(f_k)_{k=0}^{\infty} \subset W_{1,j}^{(0)}$ be a convergent sequence in Banach space $V_{1,j+1}^{(0)}$. Then $f_k \rightarrow g$ as $k \rightarrow \infty$, where $g \in V_{1,j+1}^{(0)}$. Since $P_{1,j}^{(0)}$ is continuous $P_{1,j}^{(0)}f_k \rightarrow P_{1,j}^{(0)}g$ as $k \rightarrow \infty$. Now $P_{1,j}^{(0)}f_k = 0$ for all $k \in \mathbb{N}$ and hence $P_{1,j}^{(0)}g = 0$. Furthermore $g = g - P_{1,j}^{(0)}g \in W_{1,j}^{(0)}$. Hence $W_{1,j}^{(0)}$ is closed. □

Definition 7.9. When $j \in \mathbb{Z}$ define operator $Q_{1,j}^{(0)} : C_0(\mathbb{R}, K) \rightarrow W_{1,j}^{(0)}$ by $Q_{1,j}^{(0)} := P_{1,j+1}^{(0)} - P_{1,j}^{(0)}$.

It follows that $V_{1,j+1}^{(0)} = V_{1,j}^{(0)} \dot{+} W_{1,j}^{(0)}$ for all $j \in \mathbb{Z}$. It follows from Lemma 7.5 and Equation (46) that operator $Q_{1,j}^{(0)} : C_0(\mathbb{R}, K) \rightarrow W_{1,j}^{(0)}$ is a continuous projection onto $W_{1,j}^{(0)}$ for all $j \in \mathbb{Z}$. If $x = v + w \in V_{1,j+1}^{(0)}$, $v \in V_{1,j}^{(0)}$, and $w \in W_{1,j}^{(0)}$ we have $P_{1,j}^{(0)}x = v$ and $Q_{1,j}^{(0)}x = w$. Consequently

$$\forall j' \in \mathbb{Z}, f \in V_{1,j'}^{(0)} : j' \leq j \implies Q_{1,j}^{(0)}f = 0, \quad (47)$$

$$\forall j' \in \mathbb{Z}, f \in W_{1,j'}^{(0)} : j' \neq j \implies Q_{1,j}^{(0)}f = 0, \quad (48)$$

and

$$\forall j' \in \mathbb{Z}, f \in W_{1,j'}^{(0)} : j' \geq j \implies P_{1,j}^{(0)}f = 0. \quad (49)$$

Theorem 7.10. *Let $j \in \mathbb{Z}$. Then*

$$W_{1,j}^{(0)} = \left\{ \sum_{k \in \mathbb{Z}} \mathbf{a}[k] \psi_{j,k} : \mathbf{a} \in c_0(\mathbb{Z}, K) \right\}. \quad (50)$$

Proof. Use Lemmas 3.7 and 3.9. The proof Equation (50) is similar to the beginning of the proof of [7, theorem 2.4]. □

It follows from Theorem 7.6 that

$$f = P_{1,j_0}^{(0)} f + \sum_{j=j_0}^{\infty} Q_{1,j}^{(0)} f$$

for all $f \in C_0(\mathbb{R}, K)$ where $j_0 \in \mathbb{Z}$. If $f \in C_0(\mathbb{R}, K)$ and

$$f = v + \sum_{j=j_0}^{\infty} w_j$$

where $v \in V_{j_0}$ and $w_j \in W_{1,j}^{(0)}$ for all $j \in \mathbb{Z}$, $j \geq j_0$, it follows from Equations (47), (48), and (49) that $v = P_{1,j_0}^{(0)} f$ and $W_{1,j}^{(0)} = Q_{1,j}^{(0)} f$ for all $j \in \mathbb{Z}$, $j \geq j_0$.

Lemma 7.11. *We have*

$$Q_{1,j}^{(0)} f = \sum_{k \in \mathbb{Z}} \langle \tilde{\psi}_{j,k}, f \rangle \psi_{j,k} \quad (51)$$

for all $j \in \mathbb{Z}$ and $f \in C_0(\mathbb{R}, K)$. The series in Equation (51) converges unconditionally.

Proof. Since $Q_{1,j}^{(0)} f \in W_{1,j}^{(0)}$ it follows that

$$Q_{1,j}^{(0)} f = \sum_{k \in \mathbb{Z}} a_k \psi_{j,k}$$

where $(a_k)_{k \in \mathbb{Z}} \subset K$. Furthermore, $f = P_{1,j_0}^{(0)} f + \sum_{j=j_0}^{\infty} Q_{1,j}^{(0)} f$ and it follows that $\langle \tilde{\psi}_{j,k}, f \rangle = \langle \tilde{\psi}_{j,k}, Q_{1,j}^{(0)} f \rangle = a_k$ for all $k \in \mathbb{Z}$. As $\tilde{\psi}$ is a finite linear combination of functionals $\delta(2 \cdot -k)$, $k \in \mathbb{Z}$, it follows from Lemmas 3.14 and 3.7 that the series in Equation (51) converges unconditionally. Hence the theorem is true. \square

Definition 7.12. Define

$$W_{1,b,j}^{(0)} = \begin{cases} V_{1,j}^{(0)}; & b = 0 \\ W_{1,j}^{(0)}; & b = 1 \end{cases}$$

for all $j \in \mathbb{Z}$ and $b \in \{0, 1\}$.

Definition 7.13. Define operators $Q_{1,b,j}^{(0)} : C_0(\mathbb{R}, K) \rightarrow W_{1,b,j}^{(0)}$ by

$$Q_{1,b,j}^{(0)} = \begin{cases} P_{1,j}^{(0)}; & b = 0 \\ Q_{1,j}^{(0)}; & b = 1 \end{cases}$$

for all $j \in \mathbb{Z}$, $b \in \{0, 1\}$.

7.3 Multivariate MRA of $C_0(\mathbb{R}^n, K)$

We shall assume that $n \in \mathbb{Z}_+$ throughout this subsection. Unless otherwise stated, we shall assume that the same value of n is used throughout this subsection. By Lemma 4.14 function $\varphi^{[n]}$ belongs to $C_0(\mathbb{R}^n, K)$.

Definition 7.14. Define

$$V_{n,j}^{\otimes} := \text{n.s.} \bigotimes_{k=1}^n V_{1,j}^{(0)}, \quad j \in \mathbb{Z}.$$

The injective tensor product respects subspaces and it follows from Theorem 4.14 that $V_{n,j}^{(0)}$ is a closed subspace of

$$\bigotimes_{k=1}^n C_0(\mathbb{R}, K)$$

for each $j \in \mathbb{Z}$. It follows from Theorem 4.14 that $V_{n,j}^{\otimes}$ is a closed subspace of $C_0(\mathbb{R}^n, K)$, too, for all $j \in \mathbb{Z}$. Since $V_{1,j}^{(0)} \subset_{\text{c.s.}} V_{1,j+1}^{(0)}$ for all $j \in \mathbb{Z}$ it follows that $V_{n,j}^{\otimes} \subset_{\text{c.s.}} V_{n,j+1}^{\otimes}$ for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$.

Lemma 7.15. *Let $j \in \mathbb{Z}$ and $f \in C_0(\mathbb{R}^n, K)$. Series*

$$s := \sum_{\mathbf{k} \in \mathbb{Z}^n} f\left(\frac{\mathbf{k}}{2^j}\right) \varphi_{j,\mathbf{k}}^{[n]}$$

converges unconditionally in $V_{n,j}^{(0)}$ and $\|s\|_{\infty} \leq N_{\text{cover}}(\varphi^{[n]}) \|\varphi^{[n]}\|_{\infty} \|f\|_{\infty}$.

Proof. This is a consequence of Lemma 3.14 and Lemma 3.7. \square

Theorem 7.16. *Let $j \in \mathbb{Z}$. The set $\{\varphi_{j,\mathbf{k}}^{[n]} : \mathbf{k} \in \mathbb{Z}^n\}$ is an unconditional basis of $V_{n,j}^{(0)}$ and $V_{n,j}^{(0)} = V_{n,j}^{\otimes}$.*

Proof. Space $V_{n,j}^{\otimes}$ is a closed subspace of $C_0(\mathbb{R}^n, K)$. Hence by Lemma 7.15 we have $V_{n,j}^{(0)} \subset V_{n,j}^{\otimes}$. By Lemma 4.2 the sequence $(\varphi_{j,\sigma_{\text{sq}}^{[n]}(k)}^{[n]})_{k=0}^{\infty}$ is a Schauder basis of $V_{n,j}^{\otimes}$.

Let $f \in V_{n,j}^{\otimes}$. Now

$$f = \sum_{k=0}^{\infty} b_k \varphi_{j,\sigma_{\text{sq}}^{[n]}(k)}^{[n]} = \sum_{k=0}^{\infty} f\left(\frac{\sigma_{\text{sq}}^{[n]}(k)}{2^j}\right) \varphi_{j,\sigma_{\text{sq}}^{[n]}(k)}^{[n]} \quad (52)$$

We have $V_{n,j}^{\otimes} \subset C_0(\mathbb{R}^n, K)$. Hence by Lemma 7.15 the series in Equation (52) converge unconditionally. Consequently the set $\{\varphi_{j,\mathbf{k}}^{[n]} : \mathbf{k} \in \mathbb{Z}^n\}$ is an unconditional basis of $V_{n,j}^{\otimes}$. As $f \in V_{n,j}^{\otimes}$ was arbitrary it follows from Lemma 3.14 that $V_{n,j}^{\otimes} \subset V_{n,j}^{(0)}$ and hence $V_{n,j}^{(0)} = V_{n,j}^{\otimes}$. \square

By Lemma 3.9 function $\iota_{n,j}^{(0)} : c_0(\mathbb{Z}^n, K) \rightarrow V_{n,j}^{(0)}$ defined by

$$\iota_{n,j}^{(0)}(\mathbf{a}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi_{j,\mathbf{k}}^{[n]}$$

for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$ is a topological isomorphism from $c_0(\mathbb{Z}^n, K)$ onto $V_{n,j}^{(0)}$ and

$$\|\mathbf{a}\|_{\infty} \leq \left\| \iota_{n,j}^{(0)}(\mathbf{a}) \right\|_{\infty} \leq N_{\text{cover}}(\varphi^{[n]}) \|\varphi^{[n]}\|_{\infty} \|\mathbf{a}\|_{\infty} \quad (53)$$

for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$ and $j \in \mathbb{Z}$. As a consequence of Theorem 6.4 we get the following theorem.

Theorem 7.17. *We have*

$$\bigcap_{j \in \mathbb{Z}} V_{n,j}^{(0)} = \{0\}.$$

Definition 7.18. When $j \in \mathbb{Z}$ define operator $P_{n,j}^{(0)} : C_0(\mathbb{R}^n, K) \rightarrow V_{n,j}^{(0)}$ by

$$P_{n,j}^{(0)} f = \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{\varphi}_{j,\mathbf{k}}^{[n]}, f \rangle \varphi_{j,\mathbf{k}}^{[n]}$$

for all $f \in C_0(\mathbb{R}^n, K)$.

It follows from Lemma 7.15 $P_{n,j}^{(0)}$, $j \in \mathbb{Z}$ are operators and uniformly bounded by

$$\|P_{n,j}^{(0)}\| \leq N_{\text{cover}} \left(\varphi^{[n]} \right) \|\varphi^{[n]}\|_{\infty} \quad (54)$$

for all $j \in \mathbb{Z}$. Operator $P_{n,j}^{(0)}$ is a projection onto $V_{n,j}^{(0)}$ for each $j \in \mathbb{Z}$.

Lemma 7.19. *We have $P_{n,j}^{(0)} = P_{n,j}^{(0)} \circ P_{n,j'}^{(0)}$ for all $j, j' \in \mathbb{Z}$, $j' \geq j$.*

Proof. The proof is similar to the proof of Lemma 7.5. \square

We get the following theorem as a direct consequence of Theorem 6.15.

Theorem 7.20. *There exist $c_1 \in \mathbb{R}_+$ and $c_2 \in \mathbb{R}_+$ so that*

$$\|f - P_{n,j}^{(0)}f\|_{\infty} \leq c_1 \omega(f; 2^{-j}c_2)$$

for all $f \in C_0(\mathbb{R}^n)$ and $j \in \mathbb{Z}$.

We get the following theorem as a direct consequence of Theorem 6.16.

Theorem 7.21. *Let $f \in C_0(\mathbb{R}^n)$. Now*

$$\lim_{j \rightarrow \infty} \|f - P_{n,j}^{(0)}f\|_{\infty} = 0.$$

We get the following lemma as a direct consequence of Lemma 6.22.

Lemma 7.22. *Let $t \in \mathbb{R}_+$. There exists a constant $c_1 \in \mathbb{R}_+$, which may depend on t , so that*

$$\forall f \in C_0(\mathbb{R}^n) : \omega(P_{n,j}^{(0)}f; 2^{-j}t) \leq c_1 \omega(f; 2^{-j}\sqrt{n}).$$

We prove now a general result on the tensor products of the function space $C_0(\mathbb{R}, K)$ with itself.

Theorem 7.23. *We have*

$$\bigotimes_{j=1}^n \hat{\otimes}_{\varepsilon} C_0(\mathbb{R}, K) =_{\text{n.s.}} C_0(\mathbb{R}^n, K).$$

Proof. Define

$$F^{[n]} :=_{\text{n.s.}} \bigotimes_{j=1}^n \hat{\otimes}_{\varepsilon} C_0(\mathbb{R}, K).$$

By Lemma 4.14 we have $F^{[n]} \subset C_0(\mathbb{R}^n, K)$. Let φ be the Deslauriers-Dubuc scaling function of some degree m and $\varphi^{[n]}$ the n -dimensional tensor product mother scaling function generated by φ . Let spaces $V_{1,j}^{(0)}$, $j \in \mathbb{Z}$, belong to the interpolating multiresolution analysis generated by φ and spaces $V_{n,j}^{(0)}$, $j \in \mathbb{Z}$, be the corresponding tensor product spaces. Let $f \in C_0(\mathbb{R}^n, K)$. It follows from Theorem 7.21 that $P_{n,j}^{(0)}f \rightarrow f$ as $j \rightarrow \infty$ (convergence in the supremum norm). Now $P_{n,j}^{(0)}f \in V_{n,j}^{(0)} \subset F^{[n]}$ for all $j \in \mathbb{Z}$. Hence $f \in \overline{F^{[n]}}$, where we hold $F^{[n]}$ as a subspace of the Banach space $C_0(\mathbb{R}^n, K)$. Since $F^{[n]}$ is a Banach space $\overline{F^{[n]}} =_{\text{n.s.}} F^{[n]}$. Hence $F^{[n]} =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. \square

Definition 7.24. Define

$$W_{n,\mathbf{s},j}^{(0)} :=_{\text{n.s.}} \bigotimes_{k=1}^n \hat{\otimes}_{\varepsilon} W_{1,\mathbf{s}[k],j}^{(0)}$$

where $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0, 1\}^n$.

Note that $W_{n, \mathbf{0}_n, j}^{(0)} =_{\text{n.s.}} V_{n, j}^{(0)}$.

Definition 7.25. Let $s \in \{0, 1\}^n$. Define operator $Q_{n, \mathbf{s}, j}^{(0)} : C_0(\mathbb{R}, K) \rightarrow W_{n, \mathbf{s}, j}^{(0)}$ by

$$Q_{n, \mathbf{s}, j}^{(0)} = \bigotimes_{k=1}^n Q_{1, \mathbf{s}[k], j}^{(0)}.$$

By Lemma 4.20 operator $Q_{n, \mathbf{s}, j}^{(0)}$ is a continuous projection onto space $W_{n, \mathbf{s}, j}^{(0)}$ for each $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, and $\mathbf{s} \in \{0, 1\}^n$.

Lemma 7.26. *We have*

$$Q_{n, \mathbf{s}, j}^{(0)} f = \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}, f \rangle \psi_{\mathbf{s}, j, \mathbf{k}}^{[n]}$$

for all $f \in C_0(\mathbb{R}^n, K)$, $\mathbf{s} \in \{0, 1\}^n$, and $j \in \mathbb{Z}$. Here the series converges unconditionally and $(\langle \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}, f \rangle)_{\mathbf{k} \in \mathbb{Z}^n} \in c_0(\mathbb{Z}^n, K)$.

Proof. Define operators $T_{n, \mathbf{s}', j'} : C_0(\mathbb{R}^n, K) \rightarrow W_{n, \mathbf{s}', j'}^{(0)}$, $j' \in \mathbb{Z}$, $\mathbf{s}' \in \{0, 1\}^n$, by

$$T_{n, \mathbf{s}', j'} g := \sum_{\mathbf{k} \in \mathbb{Z}^n} \left\langle \tilde{\psi}_{\mathbf{s}', j', \mathbf{k}}^{[n]}, g \right\rangle \psi_{\mathbf{s}', j', \mathbf{k}}^{[n]} \quad (55)$$

for all $g \in C_0(\mathbb{R}^n, K)$. Functional $\tilde{\psi}_{\mathbf{s}'}^{[n]}$ is a finite linear combination of functionals $\delta(2 \cdot -\mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^n$. Hence by Lemma 3.14 we have $(\langle \tilde{\psi}_{\mathbf{s}', j', \mathbf{k}}^{[n]}, g \rangle)_{\mathbf{k} \in \mathbb{Z}^n} \in c_0(\mathbb{Z}^n, K)$ and there exists $c \in \mathbb{R}_+$ so that $\left\| (\langle \tilde{\psi}_{\mathbf{s}', j', \mathbf{k}}^{[n]}, g \rangle)_{\mathbf{k} \in \mathbb{Z}^n} \right\|_\infty \leq c \|g\|_\infty$ for all $g \in C_0(\mathbb{R}^n, K)$. By Lemma 3.7 the series in Equation (55) converges unconditionally in $C_0(\mathbb{R}^n, K)$ and there exists $c' \in \mathbb{R}_+$ so that $\|T_{n, \mathbf{s}', j'} g\|_\infty \leq c' \|g\|_\infty$ for all $g \in C_0(\mathbb{R}^n, K)$.

Let $g \in V_{n, j'}^{(0)}$ for some $j' \in \mathbb{Z}$. Now $g = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi_{j', \mathbf{k}}^{[n]}$ where $\mathbf{a}[\mathbf{k}] \in c_0(\mathbb{Z}^n, K)$ and

$$Q_{n, \mathbf{s}, j}^{(0)} g = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] Q_{n, \mathbf{s}, j}^{(0)} \varphi_{j', \mathbf{k}}^{[n]} = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \bigotimes_{k=1}^n Q_{1, \mathbf{s}[k], j}^{(0)} \varphi_{j', \mathbf{k}[k]}.$$

We also have

$$Q_{1, t, j}^{(0)} \varphi_{j', m} = \sum_{l \in \mathbb{Z}} \left\langle \tilde{\zeta}_{t, j, l}, \varphi_{j', m} \right\rangle \zeta_{t, j, l}$$

for each $t \in \{0, 1\}$ and $m \in \mathbb{Z}$ and

$$\begin{aligned} \bigotimes_{k=1}^n Q_{1, \mathbf{s}[k], j}^{(0)} \varphi_{j', \mathbf{k}[k]} &= \bigotimes_{k=1}^n \sum_{l \in \mathbb{Z}} \left\langle \tilde{\zeta}_{\mathbf{s}[k], j, l}, \varphi_{j', \mathbf{k}[k]} \right\rangle \zeta_{\mathbf{s}[k], j, l} \\ &= \sum_{\ell \in \mathbb{Z}^n} \bigotimes_{k=1}^n \left\langle \tilde{\zeta}_{\mathbf{s}[k], j, \ell[k]}, \varphi_{j', \mathbf{k}[k]} \right\rangle \zeta_{\mathbf{s}[k], j, \ell[k]} \\ &= \sum_{\ell \in \mathbb{Z}^n} \left\langle \bigotimes_{k=1}^n \tilde{\zeta}_{\mathbf{s}[k], j, \ell[k]}, \bigotimes_{k=1}^n \varphi_{j', \mathbf{k}[k]} \right\rangle \bigotimes_{k=1}^n \zeta_{\mathbf{s}[k], j, \ell[k]} \\ &= \sum_{\ell \in \mathbb{Z}^n} \left\langle \tilde{\psi}_{\mathbf{s}, j, \ell}^{[n]}, \varphi_{j', \mathbf{k}}^{[n]} \right\rangle \psi_{\mathbf{s}, j, \ell}^{[n]} = T_{n, \mathbf{s}, j} \varphi_{j', \mathbf{k}}^{[n]} \end{aligned}$$

where the second equality follows from Lemma 4.26. Thus

$$Q_{n,s,j}^{(0)}g = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}]T_{n,s,j}\varphi_{j',\mathbf{k}}^{[n]} = T_{n,s,j} \left(\sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\varphi_{j',\mathbf{k}}^{[n]} \right) = T_{n,s,j}g.$$

Let $f \in C_0(\mathbb{R}^n, K)$. Now $P_{n,j'}^{(0)}f \rightarrow f$ as $j' \rightarrow \infty$ and $Q_{n,s,j}^{(0)}(P_{n,j'}^{(0)}f) = T_{n,s,j}(P_{n,j'}^{(0)}f)$ for all $j' \in \mathbb{Z}$. By continuity of operators $Q_{n,s,j}^{(0)}$ and $T_{n,s,j}$ we have $Q_{n,s,j}^{(0)}f = T_{n,s,j}f$. \square

It follows that

$$P_{n,j}^{(0)} = Q_{n,0_n,j}^{(0)} = \bigotimes_{k=1}^n P_{1,j}^{(0)}.$$

When $\mathbf{s} \in \{0,1\}^n$ it follows from Lemmas 3.7 and 7.26 that operators $Q_{n,s,j}^{(0)}$, $j \in \mathbb{Z}$, are uniformly bounded by

$$\|Q_{n,s,j}^{(0)}\| \leq N_{\text{cover}} \left(\psi_{\mathbf{s}}^{[n]} \right) \|\psi_{\mathbf{s}}^{[n]}\|_{\infty} \|\tilde{\psi}_{\mathbf{s}}^{[n]}\|$$

for all $j \in \mathbb{Z}$. We also have $Q_{n,s,j}^{(0)} = Q_{n,s,j}^{(0)} \circ P_{n,j'}^{(0)}$ for all $j, j' \in \mathbb{Z}$, $j' > j$ and $\mathbf{s} \in \{0,1\}^n$.

Definition 7.27. Define

$$W_{n,j}^{(0)} :=_{\text{n.s.}} \sum_{\mathbf{s} \in J_+(n)} W_{n,s,j}^{(0)}$$

for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$.

By Lemma 4.4 we have $W_{n,j}^{(0)} \subset_{\text{c.s.}} C_0(\mathbb{R}^n, K)$.

Definition 7.28. Define operator $Q_{n,j}^{(0)} : C_0(\mathbb{R}, K) \rightarrow W_{n,j}^{(0)}$ by

$$Q_{n,j}^{(0)} = \sum_{\mathbf{s} \in J_+(n)} Q_{n,s,j}^{(0)}.$$

Theorem 7.29. Let $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$. Then

$$W_{n,s,j}^{(0)} = \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}]\psi_{\mathbf{s},j,\mathbf{k}}^{[n]} : \mathbf{a} \in c_0(\mathbb{Z}^n, K) \right\}$$

where the series converges unconditionally.

Proof. Now $\psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \in W_{n,s,j}^{(0)}$ for each $\mathbf{k} \in \mathbb{Z}^n$. Suppose that $(a_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^n} \in c_0(\mathbb{Z}^n, K)$. By Lemma 3.7 the series $t := \sum_{\mathbf{k} \in \mathbb{Z}^n} \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} a_{\mathbf{k}}$ converges unconditionally in $C_0(\mathbb{R}^n, K)$. As $W_{n,s,j}^{(0)} \subset_{\text{c.s.}} C_0(\mathbb{R}^n, K)$ it follows that $t \in W_{n,s,j}^{(0)}$.

Let $f \in W_{n,s,j}^{(0)}$. Now

$$f = Q_{n,s,j}^{(0)}f = \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]}, f \rangle \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}$$

and $(\langle \tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]}, f \rangle)_{\mathbf{k} \in \mathbb{Z}^n} \in c_0(\mathbb{Z}^n, K)$. \square

By Lemma 3.9 function $\eta_{n,\mathbf{s},j}^{(0)} : c_0(\mathbb{Z}^n, K) \rightarrow W_{n,\mathbf{s},j}^{(0)}$ defined by

$$\eta_{n,\mathbf{s},j}^{(0)}(\mathbf{a}) := \sum_{\mathbf{k} \in \mathbb{Z}} \mathbf{a}[\mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}$$

for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$ is a topological isomorphism from $c_0(\mathbb{Z}^n, K)$ onto $W_{n,\mathbf{s},j}^{(0)}$ and

$$\|\mathbf{a}\|_\infty \leq \left\| \eta_{n,\mathbf{s},j}^{(0)}(\mathbf{a}) \right\|_\infty \leq N_{\text{cover}} \left(\psi_{\mathbf{s}}^{[n]} \right) \left\| \psi_{\mathbf{s}}^{[n]} \right\|_\infty \|\mathbf{a}\|_\infty \quad (56)$$

for all $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$, $\mathbf{s} \in \{0, 1\}^n$, and $j \in \mathbb{Z}$.

Lemma 7.30. $Q_{n,j}^{(0)} = P_{n,j+1}^{(0)} - P_{n,j}^{(0)}$.

Proof. We have $P_{n,j}^{(0)} = Q_{n,\mathbf{0}_n,j}^{(0)}$ and

$$P_{n,j+1}^{(0)} = \bigotimes_{j=1}^n P_{1,j+1}^{(0)} = \bigotimes_{j=1}^n (P_{1,j}^{(0)} + Q_{1,j}^{(0)}) = \sum_{\mathbf{s} \in \{0,1\}^n} \bigotimes_{j=1}^n Q_{1,\mathbf{s}[j],j}^{(0)} = \sum_{\mathbf{s} \in \{0,1\}^n} Q_{n,\mathbf{s},j}^{(0)}.$$

Consequently the lemma is true. \square

It follows that the operators $Q_{n,j}^{(0)}$, $j \in \mathbb{Z}$, are uniformly bounded by

$$\left\| Q_{n,j}^{(0)} \right\| \leq 2N_{\text{cover}} \left(\varphi^{[n]} \right) \left\| \varphi^{[n]} \right\|_\infty \quad (57)$$

for all $j \in \mathbb{Z}$.

Lemma 7.31. *Let $j \in \mathbb{Z}$. Then*

- (i) $\forall \mathbf{s} \in \{0, 1\}^n, \mathbf{t} \in \{0, 1\}^n : \mathbf{s} \neq \mathbf{t} \implies \forall f \in W_{n,\mathbf{s},j}^{(0)} : Q_{n,\mathbf{t},j}^{(0)} f = 0$
- (ii) $\forall j' \in \mathbb{Z}, f \in V_{n,j',j}^{(0)} : j' \leq j \implies Q_{n,j',j}^{(0)} f = 0$
- (iii) $\forall j' \in \mathbb{Z}, f \in W_{n,j',j}^{(0)} : j' \neq j \implies Q_{n,j',j}^{(0)} f = 0$
- (iv) $\forall j' \in \mathbb{Z}, f \in W_{n,j',j}^{(0)} : j' \geq j \implies P_{n,j',j}^{(0)} f = 0$.

Proof.

- (i) Let $\mathbf{s}, \mathbf{t} \in \{0, 1\}^n$, $\mathbf{s} \neq \mathbf{t}$, and $f \in W_{n,\mathbf{s},j}^{(0)}$. Now

$$f = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}$$

where $\mathbf{a} \in c_0(\mathbb{Z}^n, K)$ and

$$\begin{aligned} Q_{n,\mathbf{t},j}^{(0)} f &= \sum_{\ell \in \mathbb{Z}^n} \left\langle \tilde{\psi}_{\mathbf{t},j,\ell}^{[n]}, f \right\rangle \psi_{\mathbf{t},j,\ell}^{[n]} = \sum_{\ell \in \mathbb{Z}^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \left\langle \tilde{\psi}_{\mathbf{t},j,\ell}^{[n]}, \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \right\rangle \psi_{\mathbf{t},j,\ell}^{[n]} \\ &= \sum_{\ell \in \mathbb{Z}^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \delta_{\mathbf{t},\mathbf{s}} \delta_{\ell,\mathbf{k}} \psi_{\mathbf{t},j,\ell}^{[n]} = 0 \end{aligned}$$

where the third equality follows from Equation (30).

(ii)-(iv) The proofs are similar to the proofs in Lemma 6.12.

□

As $V_{1,j+1}^{(0)} =_{\text{n.s.}} V_{1,j}^{(0)} \dot{+} W_{1,j}^{(0)}$ it follows from Corollary 4.7, Lemma 7.31, and Equations (47), (48), and (49) that

$$V_{n,j+1}^{(0)} =_{\text{n.s.}} V_{n,j}^{(0)} \dot{+} W_{n,j}^{(0)} \quad (58)$$

for all $j \in \mathbb{Z}$.

It follows from Theorem 7.21 that

$$f = P_{n,j_0}^{(0)} f + \sum_{j=j_0}^{\infty} Q_{n,j}^{(0)} f \quad (59)$$

for all $f \in C_0(\mathbb{R}^n, K)$. If

$$f = v + \sum_{j=j_0}^{\infty} w_j$$

where $v \in V_{n,j_0}^{(0)}$ and $w_j \in W_{n,j}^{(0)}$ for all $j \in \mathbb{Z}$, $j \geq j_0$, it follows from Lemmas 7.30 and 7.31 that $v = P_{n,j_0}^{(0)} f$ and $w_j = Q_{n,j}^{(0)} f$ for all $j \in \mathbb{Z}$, $j \geq j_0$.

Definition 7.32. Let $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$. Let $E =_{\text{n.s.}} C_u(\mathbb{R}^n, K)$ or $E =_{\text{n.s.}} C_0(\mathbb{R}^n, K)$. Define

$$V_{n,j}^E :=_{\text{n.s.}} \begin{cases} V_{n,j}^{(u)}; & E = C_u(\mathbb{R}^n, K) \\ V_{n,j}^{(0)}; & E = C_0(\mathbb{R}^n, K), \end{cases}$$

$$W_{n,j}^E :=_{\text{n.s.}} \begin{cases} W_{n,j}^{(u)}; & E =_{\text{n.s.}} C_u(\mathbb{R}^n, K) \\ W_{n,j}^{(0)}; & E =_{\text{n.s.}} C_0(\mathbb{R}^n, K), \end{cases}$$

and

$$W_{n,s,j}^E :=_{\text{n.s.}} \begin{cases} W_{n,s,j}^{(u)}; & E =_{\text{n.s.}} C_u(\mathbb{R}^n, K) \\ W_{n,s,j}^{(0)}; & E =_{\text{n.s.}} C_0(\mathbb{R}^n, K). \end{cases}$$

Theorem 7.33. Let $n \in \mathbb{Z}_+$ and $j_0 \in \mathbb{Z}$. Let $E = C_u(\mathbb{R}^n, K)$ or $E = C_0(\mathbb{R}^n, K)$. Suppose that the mother scaling function φ is Lipschitz continuous and let $\varphi^{[n]}$ be the tensor product mother scaling function generated by φ . Then

$$V_{n,j_0}^E \dot{+} \sum_{j=j_0}^{\infty} W_{n,j}^E \neq E.$$

Proof. Let

$$A := V_{n,j_0}^E \dot{+} \sum_{j=j_0}^{\infty} W_{n,j}^E.$$

Define function $f \in C_0(\mathbb{R}, K)$ by

$$f(x) := \begin{cases} \sqrt{x}; & x \in [0, 1] \\ 0; & x \leq 0 \text{ or } x \geq 2 \\ -x + 2; & x \in [1, 2] \end{cases}$$

and function $f^{[n]} \in C_0(\mathbb{R}^n, K)$ by

$$f^{[n]} := \bigotimes_{k=1}^n f.$$

Functions f and $f^{[n]}$ are not Lipschitz continuous. As φ is Lipschitz continuous all the functions $P_{n,j}^{(0)}g$ are Lipschitz continuous for each $g \in C_0(\mathbb{R}^n, K)$. A is a locally convex space with an inductive limit topology. It follows from [39, theorem 6.2] that the space A is complete.

Suppose that A would be equal to E as a set. Then we would have $f^{[n]} \in A$. It follows from the definition of the locally convex direct sum that

$$f^{[n]} \in V_{n,j_0}^E \dot{+} \sum_{j=j_0}^{j_1} W_{n,j}^E$$

for some $j_1 \in \mathbb{Z}$, $j_1 \geq j_0$. Now $f^{[n]} = P_{n,j_1+1}^{(0)}f^{[n]}$. It follows that $f^{[n]}$ is Lipschitz continuous, which is a contradiction. Hence A is not equal to E as a set. \square

However, it follows from Equations (58) and (59) that

$$C_0(\mathbb{R}^n, K) =_{\text{n.s.}} \text{clos} \left(\bigcup_{l=j_0}^{\infty} \left(V_{n,j_0}^{(0)} \dot{+} \sum_{j=j_0}^l W_{n,j}^{(0)} \right) \right)$$

and by Lemma 6.13 and Equation (42) we have

$$C_u(\mathbb{R}^n, K) =_{\text{n.s.}} \text{clos} \left(\bigcup_{l=j_0}^{\infty} \left(V_{n,j_0}^{(u)} \dot{+} \sum_{j=j_0}^l W_{n,j}^{(u)} \right) \right).$$

8 Interpolating Dual MRA

We shall have $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$ throughout this section. We shall also have $E = C_u(\mathbb{R}^n, K)$ or $E = C_0(\mathbb{R}^n, K)$.

8.1 General

Definition 8.1. When $j \in \mathbb{Z}$ define

$$\begin{aligned} \tilde{V}_{n,j} &:=_{\text{n.s.}} \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{\varphi}_{j,\mathbf{k}}^{[n]} : \mathbf{d} \in l^1(\mathbb{Z}^n, K) \right\} \\ \|\tilde{f}|_{\tilde{V}_{n,j}}\| &:= \|\tilde{f}|_{C_u(\mathbb{R}^n, K)^*}\| = \|\tilde{f}|_{C_0(\mathbb{R}^n, K)^*}\|, \quad \tilde{f} \in \tilde{V}_{n,j}. \end{aligned}$$

By Lemma 3.17 we may identify Banach space $\tilde{V}_{n,j}$ for both $E = C_u(\mathbb{R}^n, K)$ and $E = C_0(\mathbb{R}^n, K)$ for each $j \in \mathbb{Z}$. It follows also that function $\tilde{t}_{n,j} : l^1(\mathbb{Z}^n, K) \rightarrow \tilde{V}_{n,j}$ defined by

$$\tilde{t}_{n,j}(\mathbf{d}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{\varphi}_{j,\mathbf{k}}^{[n]}$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$ is an isometric isomorphism from $l^1(\mathbb{Z}^n, K)$ onto $\tilde{V}_{n,j}$. We have

$$\tilde{f} \in \tilde{V}_{n,j} \iff \tilde{f}(2 \cdot) \in \tilde{V}_{n,j+1}$$

and

$$\tilde{f} \in \tilde{V}_{n,j} \iff \tilde{f}(\cdot - 2^{-j}\mathbf{k}) \in \tilde{V}_{n,j}.$$

for all $\tilde{f} \in C_u(\mathbb{R}^n, K)^*$, $j \in \mathbb{Z}$, and $\mathbf{k} \in \mathbb{Z}^n$.

Lemma 8.2. Let $E = C_u(\mathbb{R}^n, K)$ or $E = C_0(\mathbb{R}^n, K)$. Let $j \in \mathbb{Z}$ and $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection. Let $\mathbf{d} \in l^1(J_+(n) \times \mathbb{Z}^n, K)$ and $\tilde{f} \in E^*$,

$$\tilde{f} := \sum_{\mathbf{s} \in J_+(n)} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{s}, \mathbf{k}] \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}.$$

Then

$$\|\tilde{f}|E^*\| = \lim_{p \rightarrow \infty} \sum_{\ell \in \mathbb{Z}^n} \left| \sum_{\mathbf{s} \in J_+(n)} \sum_{q=0}^p \mathbf{d}[\mathbf{s}, \sigma(q)] \tilde{g}_{\mathbf{s}, \ell - 2\sigma(q)}^{[n]} \right|.$$

Proof. Define

$$\begin{aligned} \tilde{g}_p &:= \sum_{\mathbf{s} \in J_+(n)} \sum_{q=0}^p \mathbf{d}[\mathbf{s}, \sigma(q)] \tilde{\psi}_{\mathbf{s}, j, \sigma(q)}^{[n]}, & p \in \mathbb{N} \\ I(q, \mathbf{s}) &:= \left\{ \ell \in \mathbb{Z}^n : \tilde{g}_{\mathbf{s}, \ell - 2\sigma(q)}^{[n]} \neq 0 \right\}, & q \in \mathbb{N}, \mathbf{s} \in J_+(n) \\ J(p) &:= \bigcup_{\mathbf{s} \in J_+(n)} \bigcup_{q=0}^p I(q, \mathbf{s}), & p \in \mathbb{N} \end{aligned}$$

Now

$$\|\tilde{f}\| = \lim_{p \rightarrow \infty} \|\tilde{g}_p\|$$

and

$$\begin{aligned} \tilde{g}_p &= \sum_{\mathbf{s} \in J_+(n)} \sum_{q=0}^p \sum_{\ell \in J(p)} \mathbf{d}[\mathbf{s}, \sigma(q)] \tilde{g}_{\mathbf{s}, \ell - 2\sigma(q)}^{[n]} \varphi_{j+1, \ell}^{[n]} \\ &= \sum_{\ell \in J(p)} \left(\sum_{\mathbf{s} \in J_+(n)} \sum_{q=0}^p \mathbf{d}[\mathbf{s}, \sigma(q)] \tilde{g}_{\mathbf{s}, \ell - 2\sigma(q)}^{[n]} \right) \varphi_{j+1, \ell}^{[n]} \end{aligned}$$

from which the lemma follows. \square

Definition 8.3. When $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0, 1\}^n$ define

$$\begin{aligned} \tilde{W}_{n, \mathbf{s}, j} &:=_{\text{n.s.}} \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]} : \mathbf{d} \in l^1(\mathbb{Z}^n, K) \right\} \\ \|\tilde{f}| \tilde{W}_{n, \mathbf{s}, j}\| &:= \|\tilde{f}| C_u(\mathbb{R}^n, K)^*\| = \|\tilde{f}| C_0(\mathbb{R}^n, K)^*\|, \quad \tilde{f} \in \tilde{W}_{n, \mathbf{s}, j}. \end{aligned}$$

By Lemma 3.16 function $\tilde{\eta}_{n, \mathbf{s}, j} : l^1(\mathbb{Z}^n, K) \rightarrow \tilde{W}_{n, \mathbf{s}, j}$ defined by

$$\tilde{\eta}_{n, \mathbf{s}, j}(\mathbf{d}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{d}[\mathbf{k}] \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$ is a topological isomorphism from $l^1(\mathbb{Z}^n, K)$ onto $\tilde{W}_{n, \mathbf{s}, j}$ and

$$\frac{1}{N_{\text{cover}}(\psi_{\mathbf{s}}^{[n]})} \|\mathbf{d}\|_1 \leq \|\tilde{\eta}_{n, \mathbf{s}, j}(\mathbf{d})\| \leq \|\psi_{\mathbf{s}}^{[n]}\| \|\mathbf{d}\|_1$$

for all $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$. By Lemma 8.2 we may identify the Banach space $\tilde{W}_{n, \mathbf{s}, j}$ for both $E = C_u(\mathbb{R}^n, K)$ and $E = C_0(\mathbb{R}^n, K)$ for each $\mathbf{s} \in \{0, 1\}^n$ and $j \in \mathbb{Z}$.

Definition 8.4. When $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$ define

$$\tilde{P}_{n,j}\tilde{f} := \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, \varphi_{j,\mathbf{k}}^{[n]} \rangle \tilde{\varphi}_{j,\mathbf{k}}^{[n]}$$

for all $\tilde{f} \in E^*$.

Definition 8.5. When $n \in \mathbb{Z}_+$, $\mathbf{s} \in \{0,1\}^n$, and $j \in \mathbb{Z}$ define

$$\tilde{Q}_{n,\mathbf{s},j}\tilde{f} := \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \rangle \tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]}$$

for all $\tilde{f} \in E^*$.

We have $\tilde{Q}_{n,\mathbf{0}_n,j} = \tilde{P}_{n,j}$ for all $j \in \mathbb{Z}$. When $\mathbf{s} \in \{0,1\}^n$ it follows from Lemmas 3.11 and 3.13 that operators $\tilde{Q}_{n,\mathbf{s},j}$, $j \in \mathbb{Z}$, are uniformly bounded by

$$\|\tilde{Q}_{n,\mathbf{s},j}\| \leq \|\tilde{\psi}_{\mathbf{s}}^{[n]}\| \|\psi_{\mathbf{s}}^{[n]}\|_{\infty} N_{\text{cover}}(\psi_{\mathbf{s}}^{[n]})$$

for all $j \in \mathbb{Z}$ and operators $\tilde{P}_{n,j}$, $j \in \mathbb{Z}$, uniformly bounded by

$$\|\tilde{P}_{n,j}\| \leq \|\tilde{\varphi}^{[n]}\| \|\varphi^{[n]}\|_{\infty} N_{\text{cover}}(\varphi^{[n]}) \quad (60)$$

for all $j \in \mathbb{Z}$.

Lemma 8.6.

- (i) $\forall j \in \mathbb{Z}, \mathbf{s} \in \{0,1\}^n, \mathbf{t} \in \{0,1\}^n : \mathbf{s} \neq \mathbf{t} \implies \forall \tilde{f} \in \tilde{W}_{n,\mathbf{s},j} : \tilde{Q}_{n,\mathbf{t},j}\tilde{f} = 0$.
- (ii) Operator $\tilde{Q}_{n,\mathbf{s},j}$ is a continuous projection of E^* onto $\tilde{W}_{n,\mathbf{s},j}$ for all $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$.
- (iii) $\tilde{Q}_{n,\mathbf{s},j} = \tilde{Q}_{n,\mathbf{s},j} \circ \tilde{P}_{n,j'}$ for all $j, j' \in \mathbb{Z}$, $j < j'$, and $\mathbf{s} \in \{0,1\}^n$.

Lemma 8.7. Let $j \in \mathbb{Z}$, $\mathbf{s}, \mathbf{t} \in \{0,1\}^n$, and $\mathbf{s} \neq \mathbf{t}$. Then $\tilde{W}_{n,\mathbf{s},j} \cap \tilde{W}_{n,\mathbf{t},j} = \{0\}$.

Lemma 8.8. Let $j \in \mathbb{Z}$ and $\tilde{f} \in \tilde{V}_{n,j+1}$. Then

$$\sum_{\mathbf{s} \in \{0,1\}^n} \tilde{Q}_{n,\mathbf{s},j}\tilde{f} = \tilde{f}.$$

Proof. Use Lemmas 5.13 and 5.15 and Equation (28). □

Definition 8.9. When $j \in \mathbb{Z}$ define

$$\tilde{W}_{n,j} :=_{\text{n.s.}} \sum_{\mathbf{s} \in J_+(n)} \tilde{W}_{n,\mathbf{s},j}.$$

Definition 8.10. When $j \in \mathbb{Z}$ define

$$\tilde{Q}_{n,j} := \sum_{\mathbf{s} \in J_+(n)} \tilde{Q}_{n,\mathbf{s},j}.$$

Lemma 8.11. Let $j \in \mathbb{Z}$. Then

(i) Operator $\tilde{Q}_{n,j}$ is a continuous projection of E^* onto $\tilde{W}_{n,j}$.

(ii) $\tilde{Q}_{n,j}\tilde{f} = 0$ for all $\tilde{f} \in \tilde{V}_{n,j}$.

Proof. This is a consequence of Lemma 8.6. □

Lemma 8.12. When $j \in \mathbb{Z}$ we have $\tilde{Q}_{n,j} = \tilde{P}_{n,j+1} - \tilde{P}_{n,j}$.

Proof. Use Lemmas 8.6 (iii) and 8.8. □

Lemma 8.13. When $j \in \mathbb{Z}$ we have $\tilde{V}_{n,j+1} =_{\text{n.s.}} \tilde{V}_{n,j} \dot{+} \tilde{W}_{n,j}$.

Proof. Use Lemmas 8.6 (ii), 8.7, and 8.8. □

Theorem 8.14.

(i) $\forall j \in \mathbb{Z}, \mathbf{s} \in \{0, 1\}^n, \mathbf{t} \in \{0, 1\}^n : \mathbf{s} \neq \mathbf{t} \implies \tilde{W}_{n,\mathbf{s},j} \perp W_{n,\mathbf{t},j}^E$

(ii) $\forall j_1, j_2 \in \mathbb{Z} : j_1 \neq j_2 \implies \tilde{V}_{n,j_1} \perp W_{n,j_2}^E$

(iii) $\forall j_1, j_2 \in \mathbb{Z} : j_1 \neq j_2 \implies \tilde{W}_{n,j_1} \perp V_{n,j_2}^E$

(iv) $\forall j_1, j_2 \in \mathbb{Z} : j_1 \neq j_2 \implies \tilde{W}_{n,j_1} \perp W_{n,j_2}^E$

Proof. Let $j \in \mathbb{Z}, \mathbf{s}, \mathbf{t} \in \{0, 1\}^n$, and $\mathbf{k}, \ell \in \mathbb{Z}^n$. Proposition (i) follows from Equation (30). Propositions (ii) and (iii) are consequences of (i).

Suppose that $j_1, j_2 \in \mathbb{Z}$ and $j_1 \neq j_2$. If $j_1 > j_2$ then $W_{n,j_2}^E \subset V_{n,j_1}^E$ and $\tilde{W}_{n,j_1} \perp V_{n,j_1}^E$ and hence $\tilde{W}_{n,j_1} \perp W_{n,j_2}^E$. If $j_1 < j_2$ then $\tilde{W}_{n,j_1} \subset \tilde{V}_{n,j_2}$ and $\tilde{V}_{n,j_2} \perp W_{n,j_2}^E$ and hence $\tilde{W}_{n,j_1} \perp W_{n,j_2}^E$. So proposition (iv) is true. □

Theorem 8.15. Let $\tilde{f} \in C_0(\mathbb{R}^n, K)^*$. Then $\tilde{P}_{n,j}\tilde{f} \xrightarrow{w^*} \tilde{f}$ as $j \rightarrow \infty$.

Proof. Let $\tilde{f} \in C_0(\mathbb{R}^n, K)^*$. Suppose that $f \in C_0(\mathbb{R}^n, K)$. Then

$$\langle \tilde{P}_{n,j}\tilde{f}, P_{n,j}^{(0)}f \rangle = \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, \varphi_{j,\mathbf{k}}^{[n]} \rangle \langle \tilde{\varphi}_{j,\mathbf{k}}^{[n]}, P_{n,j}^{(0)}f \rangle = \left\langle \tilde{f}, \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{\varphi}_{j,\mathbf{k}}^{[n]}, f \rangle \varphi_{j,\mathbf{k}}^{[n]} \right\rangle = \langle \tilde{f}, P_{n,j}^{(0)}f \rangle.$$

By Equation (60) there exists $c \in \mathbb{R}_+$ so that $\|\tilde{P}_{n,j}\| \leq c$ for all $j \in \mathbb{Z}$. Consequently

$$\begin{aligned} \left| \langle \tilde{f}, f \rangle - \langle \tilde{P}_{n,j}\tilde{f}, f \rangle \right| &\leq \left| \langle \tilde{f}, f \rangle - \langle \tilde{f}, P_{n,j}^{(0)}f \rangle \right| + \left| \langle \tilde{P}_{n,j}\tilde{f}, P_{n,j}^{(0)}f \rangle - \langle \tilde{P}_{n,j}\tilde{f}, f \rangle \right| \\ &= \left| \langle \tilde{f}, f - P_{n,j}^{(0)}f \rangle \right| + \left| \langle \tilde{P}_{n,j}\tilde{f}, P_{n,j}^{(0)}f - f \rangle \right| \\ &\leq \|\tilde{f}\| \|f - P_{n,j}^{(0)}f\| + c \|\tilde{f}\| \|f - P_{n,j}^{(0)}f\| \rightarrow 0, \text{ as } j \rightarrow \infty. \end{aligned}$$

□

8.2 Tensor Product Representation of the Dual MRA

Lemma 8.16. *Banach space $C_0(\mathbb{R}^n, K)^* \hat{\otimes}_{\varepsilon^s} C_0(\mathbb{R}, K)^*$ is a closed subspace of $C_0(\mathbb{R}^{n+1}, K)^*$.*

Proof. By Theorem 7.23

$$C_0(\mathbb{R}^n, K) \hat{\otimes}_{\varepsilon} C_0(\mathbb{R}, K) = \left(\bigotimes_{k=1}^n C_0(\mathbb{R}, K) \right) \hat{\otimes}_{\varepsilon} C_0(\mathbb{R}, K) = \bigotimes_{k=1}^{n+1} C_0(\mathbb{R}, K) = C_0(\mathbb{R}^{n+1}, K).$$

Hence it follows from Equation (4) that $C_0(\mathbb{R}^n, K)^* \hat{\otimes}_{\varepsilon^s} C_0(\mathbb{R}, K)^*$ is a closed subspace of $C_0(\mathbb{R}^{n+1}, K)^*$. \square

Definition 8.17. Let $j \in \mathbb{Z}$. Define

$$\tilde{M}_{n,j} :=_{\text{n.s.}} \bigotimes_{k=1}^n \tilde{V}_{1,j}.$$

The Banach spaces $\tilde{M}_{n,j}$ are well defined since $\tilde{V}_{1,j}$ is a closed subspace of $C_0(\mathbb{R}, K)^*$. Vector space $C_0(\mathbb{R}^k, K)^* \otimes \tilde{V}_{1,j}$ is a linear subspace of $C_0(\mathbb{R}^k, K)^* \otimes C_0(\mathbb{R}, K)^*$ for all $k \in \mathbb{Z}_+$. It follows from Lemma 8.16 that $C_0(\mathbb{R}^k, K)^* \otimes \tilde{V}_{1,j}$ is a linear subspace of $C_0(\mathbb{R}^{k+1}, K)^*$ for all $k \in \mathbb{Z}_+$. Consequently $\tilde{V}_{n,j}$ is a closed subspace of $C_0(\mathbb{R}^n, K)^*$ for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$. Since $\tilde{V}_{1,j} \subset_{\text{c.s.}} \tilde{V}_{1,j+1}$ for all $j \in \mathbb{Z}$ it follows that $\tilde{V}_{n,j} \subset_{\text{c.s.}} \tilde{V}_{n,j+1}$ for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$.

Lemma 8.18. *When $j \in \mathbb{Z}$*

$$\bigotimes_{k=1}^n \tilde{V}_{1,j} =_{\text{n.s.}} \tilde{V}_{n,j}.$$

Proof. Let

$$E :=_{\text{n.s.}} \bigotimes_{k=1}^n \tilde{V}_{1,j}$$

and

$$F :=_{\text{n.s.}} \bigotimes_{k=1}^n l^1(\mathbb{Z}, K).$$

Let

$$\alpha := \bigotimes_{k=1}^n \tilde{l}_{1,j}$$

By Lemma 4.21 α is an isometric isomorphism from F onto E . Define function $\xi^{[n]}$ as in Lemma 4.25. Let $\beta := \alpha \circ (\xi^{[n]})^{-1}$. Now β is an isometric isomorphism from $l^1(\mathbb{Z}^n, K)$ onto E and $\beta(\check{\mathbf{e}}_{\mathbf{k}}) = \tilde{\varphi}_{j,\mathbf{k}}^{[n]} = \tilde{l}_{n,j}(\check{\mathbf{e}}_{\mathbf{k}})$ for all $\mathbf{k} \in \mathbb{Z}^n$. When $\mathbf{d} \in l^1(\mathbb{Z}^n, K)$ we have $\beta(\mathbf{d}) = \tilde{l}_{n,j}(\mathbf{d})$ and $\|\beta(\mathbf{d})\|_E = \|\tilde{l}_{n,j}(\mathbf{d})\|_{C_0(\mathbb{R}^n, K)^*}$. \square

Lemma 8.19. *When $j \in \mathbb{Z}$*

$$\tilde{V}_{n,j} =_{\text{n.s.}} \tilde{M}_{n,j} =_{\text{n.s.}} \bigotimes_{k=1}^n \tilde{V}_{1,j} =_{\text{n.s.}} \bigotimes_{k=1}^n \tilde{V}_{1,j}.$$

Proof. Use induction by n , metric approximation property of l^1 , and [38, prop. 7.1]. \square

Definition 8.20. Define

$$\tilde{N}_{n,\mathbf{s},j} :=_{\text{n.s.}} \bigotimes_{k=1}^n \tilde{W}_{1,\mathbf{s}[k],j}$$

where $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$.

Lemma 8.21. Let $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0,1\}^n$. Then

$$\tilde{N}_{n,\mathbf{s},j} = \text{clos}_{C_0(\mathbb{R}^n, K)^*} \text{span}\{\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} : \mathbf{k} \in \mathbb{Z}^n\}.$$

Proof. Let

$$\tilde{A}_{n,\mathbf{s},j} = \text{clos}_{C_0(\mathbb{R}^n, K)^*} \text{span}\{\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} : \mathbf{k} \in \mathbb{Z}^n\}$$

for all $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, and $\mathbf{s} \in \{0,1\}^n$. We have $\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} \in \tilde{N}_{n,\mathbf{s},j}$ for all $\mathbf{k} \in \mathbb{Z}^n$. Hence $\text{span}\{\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} : \mathbf{k} \in \mathbb{Z}^n\} \subset \tilde{N}_{n,\mathbf{s},j}$. As $\tilde{N}_{n,\mathbf{s},j}$ is a Banach space it follows that $\tilde{A}_{n,\mathbf{s},j}$ is a closed subspace of $\tilde{N}_{n,\mathbf{s},j}$.

When $n' = 1$ we have $\tilde{N}_{1,\mathbf{s},j} =_{\text{n.s.}} \tilde{V}_{1,j}$ or $\tilde{N}_{1,\mathbf{s},j} =_{\text{n.s.}} \tilde{W}_{1,j}$ and hence $\tilde{N}_{1,\mathbf{s},j} \subset_{\text{c.s.}} \tilde{A}_{1,\mathbf{s},j}$, from which it follows that $\tilde{N}_{1,\mathbf{s},j} =_{\text{n.s.}} \tilde{A}_{1,\mathbf{s},j}$. Suppose that $\tilde{N}_{n',\mathbf{s},j} = \tilde{A}_{n',\mathbf{s},j}$ for all $j \in \mathbb{Z}$, $\mathbf{s} \in \{0,1\}^{n'}$ for some $n' \in \mathbb{Z}_+$ (induction assumption). Let $\tilde{x} \in \tilde{N}_{n'+1,\mathbf{s}',j}$ for some $\mathbf{s}' \in \{0,1\}^{n'+1}$. Let $h \in \mathbb{R}_+$. Let $\mathbf{t} := s_{\text{proj}}(n', \mathbf{s}')$ and $u := \mathbf{s}'[n' + 1]$. There exists $\tilde{y} \in \tilde{N}_{n',\mathbf{t},j} \otimes_{(C_0(\mathbb{R}^{n'+1}, K)^*)} \tilde{W}_{1,u,j}$ so that $\|\tilde{x} - \tilde{y}\| < \frac{h}{2}$. Now

$$\tilde{y} = \sum_{k=1}^m \tilde{w}_k \otimes \tilde{v}_k$$

where $m \in \mathbb{N}$ and $\tilde{w}_k \in \tilde{N}_{n',\mathbf{t},j}$, $\tilde{v}_k \in \tilde{W}_{1,u,j}$ for each $k \in Z(m)$. Let $c := \max\{\|\tilde{v}_k\| : k \in Z(m)\} + 1$. There exist $\tilde{r}_k \in \text{span}\{\tilde{\psi}_{\mathbf{t},j,\mathbf{k}}^{[n']}\} : \mathbf{k} \in \mathbb{Z}^{n'}\}$ so that

$$\|\tilde{w}_k - \tilde{r}_k\| < \frac{h}{2mc}$$

for each $k \in Z(m)$.

Now

$$\tilde{r}_k = \sum_{\ell \in J} b_{k,\ell} \tilde{\psi}_{\mathbf{t},j,\ell}^{[n']}$$

where J is a finite subset of $\mathbb{Z}^{n'}$ and $b_{k,\ell} \in K$ for all $\ell \in J$ and $k \in Z(m)$. We also have

$$\tilde{v}_k = \sum_{p \in \mathbb{Z}} d_{k,p} \tilde{\zeta}_{u,j,p}$$

where $(d_{k,p})_{p=0}^\infty \in l^1(\mathbb{Z}, K)$ and $k \in Z(m)$. It follows that

$$\tilde{r}_k \otimes \tilde{v}_k = \sum_{\ell \in J} \sum_{p \in \mathbb{Z}} b_{p,\ell} d_{k,p} \tilde{\psi}_{\mathbf{s}',j,\mathbf{s}_{\text{comb}}(\ell,p)}^{[n'+1]} \in \tilde{A}_{n'+1,\mathbf{s}',j}$$

for each $k \in Z(m)$. Let

$$\tilde{z} = \sum_{k=1}^m \tilde{r}_k \otimes \tilde{v}_k.$$

Now $\tilde{z} \in \tilde{A}_{n'+1,\mathbf{s},j}$ and

$$\|\tilde{y} - \tilde{z}\| \leq \sum_{k=1}^m \|\tilde{w}_k - \tilde{r}_k\| \|\tilde{v}_k\| < \sum_{p=1}^m \frac{h}{2mc} c = \frac{h}{2}.$$

Hence $\|\tilde{x} - \tilde{z}\| \leq \|\tilde{x} - \tilde{y}\| + \|\tilde{y} - \tilde{z}\| < \frac{h}{2} + \frac{h}{2} = h$. Number $h > 0$ was arbitrary and hence $\tilde{x} \in \tilde{A}_{n'+1,\mathbf{s}',j}$. Therefore the proposition is true for $n' + 1$ and consequently for all $n \in \mathbb{Z}_+$. \square

Definition 8.22. Let $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0, 1\}^n$. Define function $\tilde{R}_{n,\mathbf{s},j} : \tilde{V}_{n,j+1} \rightarrow \tilde{N}_{n,\mathbf{s},j}$ by

$$\tilde{R}_{n,\mathbf{s},j} = \bigotimes_{k=1}^n_{(C_0(\mathbb{R}^k, K)^*, C_0(\mathbb{R}^k, K)^*)} \left(\tilde{Q}_{1,\mathbf{s}[k],j} | \tilde{V}_{1,j+1} \right)$$

Theorem 8.23. Functions $\tilde{R}_{n,\mathbf{s},j}$, $j \in \mathbb{Z}$, $\mathbf{s} \in \{0, 1\}^n$ are well defined, linear, and continuous.

Proof. Let $j \in \mathbb{Z}$ and $\mathbf{s} \in \{0, 1\}^n$. Functions $\tilde{W}_{1,b,j}^{(0)}$, $b \in \{0, 1\}$, are linear and continuous. Let $P_k = \tilde{Q}_{1,\mathbf{s}[k],j}^{(0)} | \tilde{V}_{1,j+1}^{(0)}$ for $l = 1, \dots, n$. Define the operators S_k and T_k as in Definition 4.19. Now $T_1 = S_1 = P_1$ is continuous. Suppose that T_k is continuous for some $k \in \mathbb{Z}(n-1)$. Operator T_k is a linear operator from Banach space $\tilde{V}_{k,j+1}$ into Banach space

$$\bigotimes_{m=1}^k_{(C_0(\mathbb{R}^m, K)^*)} \tilde{W}_{1,\mathbf{s}[m],j}^{(0)} =_{\text{n.s.}} \tilde{N}_{k,\text{s}_{\text{proj}}(k,\mathbf{s}),j}.$$

Operator P_{k+1} is a continuous linear operator from $\tilde{V}_{1,j+1}^{(0)}$ into $\tilde{W}_{1,\mathbf{s}[k+1],j}^{(0)}$. Subspace $\tilde{V}_{k,j+1}$ is topologically complemented in $C_0(\mathbb{R}^k, K)^*$ and subspace $\tilde{V}_{1,j+1}^{(0)}$ is topologically complemented in $C_0(\mathbb{R}, K)^*$. By Lemma 4.15 operator $S_{k+1} = T_k \otimes P_{k+1}$ is a continuous linear operator from normed vector space $\tilde{V}_{k,j+1} \otimes_{(C_0(\mathbb{R}^{k+1}, K)^*)} \tilde{V}_{1,j+1}^{(0)}$ into normed vector space $\tilde{N}_{k,\text{s}_{\text{proj}}(k,\mathbf{s}),j} \otimes_{(C_0(\mathbb{R}^{k+1}, K)^*)} \tilde{W}_{1,\mathbf{s}[k+1],j}^{(0)}$. Hence T_{k+1} is a continuous linear operator from Banach space $\tilde{V}_{k+1,j+1}$ into Banach space $\tilde{N}_{k+1,\text{s}_{\text{proj}}(k+1,\mathbf{s}),j}$. It follows that $\tilde{R}_{n,\mathbf{s},j} = T_n$ is well defined and it is a continuous linear operator from Banach space $\tilde{V}_{n,j+1}$ into Banach space $\tilde{N}_{n,\mathbf{s},j}$. \square

Lemma 8.24. Let $j \in \mathbb{Z}$, and $\mathbf{s} \in \{0, 1\}^n$. Then

$$\tilde{R}_{n,\mathbf{s},j} \tilde{f} = \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{f}, \psi_{\mathbf{s},j,\mathbf{k}}^{[n]} \rangle \tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]} \quad (61)$$

for all $\tilde{f} \in \tilde{V}_{n,j+1}$. The series in Equation (61) converges absolutely for all $\tilde{f} \in \tilde{V}_{n,j+1}$. When $\mathbf{s} \in \{0, 1\}^n$ operators $\tilde{R}_{n,\mathbf{s},j}$, $j \in \mathbb{Z}$, are uniformly bounded by

$$\left\| \tilde{R}_{n,\mathbf{s},j} \right\| \leq N_{\text{cover}(\psi_{\mathbf{s}}^{[n]})} \left\| \psi_{\mathbf{s}}^{[n]} \right\|_{\infty} \left\| \tilde{\psi}_{\mathbf{s}}^{[n]} \right\|$$

for all $j \in \mathbb{Z}$.

Proof. Define linear operators $\tilde{T}_{n,\mathbf{s}',j'} : C_0(\mathbb{R}^n, K)^* \rightarrow \tilde{W}_{n,\mathbf{s}',j'}^{(0)}$, $j' \in \mathbb{Z}$, $\mathbf{s}' \in \{0, 1\}^n$, by

$$\tilde{T}_{n,\mathbf{s}',j'} \tilde{g} := \sum_{\mathbf{k} \in \mathbb{Z}^n} \left\langle \tilde{g}, \psi_{\mathbf{s}',j',\mathbf{k}}^{[n]} \right\rangle \tilde{\psi}_{\mathbf{s}',j',\mathbf{k}}^{[n]} \quad (62)$$

for all $\tilde{g} \in C_0(\mathbb{R}^n, K)^*$. By Lemmas 3.13 and 3.11 the series in Equation (62) converges absolutely, operators $\tilde{T}_{n,\mathbf{s}',j'}$ are well-defined and continuous, and

$$\left\| \tilde{T}_{n,\mathbf{s}',j'} \right\| \leq N_{\text{cover}(\psi_{\mathbf{s}'}^{[n]})} \left\| \psi_{\mathbf{s}'}^{[n]} \right\|_{\infty} \left\| \tilde{\psi}_{\mathbf{s}'}^{[n]} \right\| \quad (63)$$

for all $j' \in \mathbb{Z}$ and $\mathbf{s}' \in \{0, 1\}^n$.

Let $\tilde{f} \in \tilde{V}_{n,j+1}$. By Lemma 8.19

$$\tilde{f} = \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \tilde{\varphi}_{j+1,\mathbf{k}}^{[n]}$$

where $\mathbf{a} \in l^1(\mathbb{Z}^n, K)$ and the series converges absolutely. Let $\sigma : \mathbb{N} \rightarrow \mathbb{Z}^n$ be a bijection. Let

$$\tilde{f}_m := \sum_{k=0}^m \mathbf{a}[\sigma(k)] \tilde{\varphi}_{j+1, \sigma(k)}^{[n]}$$

for all $m \in \mathbb{N}$. Now $\tilde{f}_m \rightarrow \tilde{f}$ as $m \rightarrow \infty$. Furthermore,

$$\begin{aligned} \tilde{R}_{n, \mathbf{s}, j} \tilde{f}_m &= \sum_{k=0}^m \mathbf{a}[\sigma(k)] \bigotimes_{l=1}^n \left(Q_{1, \mathbf{s}[l], j}^{(0)} \tilde{\varphi}_{j+1, \sigma(k)}^{[n]} \right) = \sum_{k=0}^m \mathbf{a}[\sigma(k)] \bigotimes_{l=1}^n \left(\sum_{p \in \mathbb{Z}} \langle \tilde{\varphi}_{j+1, \sigma(k)}^{[n]}[l], \zeta_{\mathbf{s}[l], j, p} \rangle \tilde{\zeta}_{\mathbf{s}[l], j, p} \right) \\ &= \sum_{k=0}^m \mathbf{a}[\sigma(k)] \sum_{p_1 \in \mathbb{Z}} \cdots \sum_{p_n \in \mathbb{Z}} \left\langle \tilde{\varphi}_{j+1, \sigma(k)}^{[n]}, \psi_{\mathbf{s}, j, (p_1, \dots, p_n)}^{[n]} \right\rangle \tilde{\psi}_{\mathbf{s}, j, (p_1, \dots, p_n)}^{[n]} \end{aligned} \quad (64)$$

Since $\psi_{\mathbf{s}, j, \mathbf{k}}^{[n]} = \psi_{\mathbf{s}}^{[n]}(2^j \cdot -\mathbf{k})$ and functions $\psi_{\mathbf{s}}^{[n]}$ are compactly supported the series in formula (64) has only finite number of nonzero terms. Hence

$$\begin{aligned} \tilde{R}_{n, \mathbf{s}, j} \tilde{f}_m &= \sum_{k=0}^m \mathbf{a}[\sigma(k)] \sum_{\mathbf{k} \in \mathbb{Z}^n} \left\langle \tilde{\varphi}_{j+1, \sigma(k)}^{[n]}, \psi_{\mathbf{s}, j, \mathbf{k}}^{[n]} \right\rangle \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]} = \sum_{\mathbf{k} \in \mathbb{Z}^n} \left\langle \sum_{k=0}^m \mathbf{a}[\sigma(k)] \tilde{\varphi}_{j+1, \sigma(k)}^{[n]}, \psi_{\mathbf{s}, j, \mathbf{k}}^{[n]} \right\rangle \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]} \\ &= \sum_{\mathbf{k} \in \mathbb{Z}^n} \left\langle \tilde{f}_m, \psi_{\mathbf{s}, j, \mathbf{k}}^{[n]} \right\rangle \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]} = \tilde{T}_{n, \mathbf{s}, j} \tilde{f}_m. \end{aligned}$$

Since operators $\tilde{R}_{n, \mathbf{s}, j}$ and $\tilde{T}_{n, \mathbf{s}, j}$ are continuous we have

$$\tilde{R}_{n, \mathbf{s}, j} \tilde{f} = \lim_{m \rightarrow \infty} \tilde{R}_{n, \mathbf{s}, j} \tilde{f}_m = \lim_{m \rightarrow \infty} \tilde{T}_{n, \mathbf{s}, j} \tilde{f}_m = \tilde{T}_{n, \mathbf{s}, j} \tilde{f}.$$

□

Lemma 8.25. *Let $j \in \mathbb{Z}$. Then*

- (i) $\forall \mathbf{s} \in \{0, 1\}^n, \tilde{f} \in \tilde{N}_{n, \mathbf{s}, j} : \tilde{R}_{n, \mathbf{s}, j} \tilde{f} = \tilde{f}$
- (ii) $\forall \mathbf{s} \in \{0, 1\}^n, \mathbf{t} \in \{0, 1\}^n, \tilde{f} \in \tilde{N}_{n, \mathbf{s}, j} : \mathbf{s} \neq \mathbf{t} \implies \tilde{R}_{n, \mathbf{t}, j} \tilde{f} = 0$
- (iii) $\forall \mathbf{s} \in J_+(n), \tilde{f} \in \tilde{V}_{n, j} : \tilde{R}_{n, \mathbf{s}, j} \tilde{f} = 0.$
- (iv) *Operator $\tilde{R}_{n, \mathbf{s}, j}$ is a projection of $\tilde{V}_{n+1, j+1}$ onto $\tilde{N}_{n, \mathbf{s}, j}$ for each $\mathbf{s} \in \{0, 1\}^n$.*

Proof.

- (i) and (ii) Let $\mathbf{s}, \mathbf{t} \in \{0, 1\}^n$ and $\tilde{f} \in \tilde{N}_{n, \mathbf{s}, j}$. By Lemma 8.21 there exists a sequence $(\tilde{f}_m)_{m=0}^\infty \subset \text{span} \left\{ \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]} : \mathbf{k} \in \mathbb{Z}^n \right\}$ so that $\tilde{f}_m \rightarrow \tilde{f}$ as $m \rightarrow \infty$ (strong convergence). Now

$$\tilde{f}_m = \sum_{\mathbf{k} \in J_m} a_{m, \mathbf{k}} \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}$$

where J_m is a finite subset of \mathbb{Z}^n and $a_{m, \mathbf{k}} \in K$ for each $\mathbf{k} \in J_m$ for all $m \in \mathbb{N}$. Furthermore,

$$\begin{aligned} \tilde{R}_{n, \mathbf{t}, j} \tilde{f}_m &= \sum_{\ell \in \mathbb{Z}^n} \left\langle \sum_{\mathbf{k} \in J_m} a_{m, \mathbf{k}} \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}, \psi_{\mathbf{t}, j, \ell}^{[n]} \right\rangle \tilde{\psi}_{\mathbf{t}, j, \ell}^{[n]} = \sum_{\ell \in \mathbb{Z}^n} \left(\sum_{\mathbf{k} \in J_m} a_{m, \mathbf{k}} \delta_{\mathbf{s}, \mathbf{t}} \delta_{\mathbf{k}, \ell} \right) \tilde{\psi}_{\mathbf{t}, j, \ell}^{[n]} \\ &= \delta_{\mathbf{s}, \mathbf{t}} \sum_{\ell \in \mathbb{Z}^n} a_{m, \ell} \tilde{\psi}_{\mathbf{t}, j, \ell}^{[n]} = \delta_{\mathbf{s}, \mathbf{t}} \tilde{f}_m. \end{aligned}$$

Since $\tilde{R}_{n,t,j}$ is continuous it follows that

$$\tilde{R}_{n,t,j}\tilde{f} = \lim_{m \rightarrow \infty} \tilde{R}_{n,s,j}\tilde{f}_m = \delta_{s,t} \lim_{m \rightarrow \infty} \tilde{f}_m = \delta_{s,t}\tilde{f}.$$

Thus both (i) and (ii) are true.

(iii) This is a consequence of (ii).

(iv) The range of operator $\tilde{R}_{n,s,j}$ is $\tilde{N}_{n,s,j}$. Hence (i) implies (iv). □

Theorem 8.26. *Let $j \in \mathbb{Z}$ and $s \in \{0, 1\}^n$. Then*

(i) *The set $\{\tilde{\psi}_{s,j,k}^{[n]} : k \in \mathbb{Z}^n\}$ is an absolutely convergent basis of Banach space $\tilde{N}_{n,s,j}$.*

(ii) $\tilde{W}_{n,s,j} =_{n.s.} \tilde{N}_{n,s,j}$.

Proof. Since $\tilde{N}_{n,s,j}$ is a closed subspace of $C_0(\mathbb{R}^n, K)^*$ it follows that $\tilde{W}_{n,s,j} \subset \tilde{N}_{n,s,j}$.

Let $\tilde{g} \in \tilde{N}_{n,s,j}$. By Lemmas 8.24 and 8.25

$$\tilde{g} = \tilde{R}_{n,s,j}\tilde{g} = \sum_{k \in \mathbb{Z}^n} \langle \tilde{g}, \psi_{s,j,k}^{[n]} \rangle \tilde{\psi}_{s,j,k}^{[n]}. \quad (65)$$

where the series converges absolutely. Thus $\{\tilde{\psi}_{s,j,k}^{[n]} : k \in \mathbb{Z}^n\}$ is an absolutely convergent basis of $\tilde{N}_{n,s,j}$ so (i) is true.

It follows from Lemma 3.13 that $(\langle \tilde{g}, \psi_{s,j,k}^{[n]} \rangle)_{k \in \mathbb{Z}^n} \in l^1(\mathbb{Z}^n, K)$. As $\tilde{g} \in \tilde{N}_{n,s,j}$ was arbitrary we have $\tilde{N}_{n,s,j} \subset_{c.s.} \tilde{W}_{n,s,j}$. Hence $\tilde{W}_{n,s,j} = \tilde{N}_{n,s,j}$ and (ii) is true. □

Lemma 8.27. *Let $j \in \mathbb{Z}$ and $s \in \{0, 1\}^n$. Then $\tilde{Q}_{n,s,j} = \tilde{R}_{n,s,j} \circ \tilde{P}_{n,j+1}$.*

Proof. Let $\tilde{f} \in E^*$. By Lemma 8.6 (ii) we have $\tilde{P}_{n,j+1}\tilde{f} \in \tilde{V}_{n,j+1}$. It follows by Lemma 8.24 that

$$\tilde{R}_{n,s,j}(\tilde{P}_{n,j+1}(\tilde{f})) = \sum_{k \in \mathbb{Z}^n} \left\langle \tilde{P}_{n,j+1}\tilde{f}, \psi_{s,j,k}^{[n]} \right\rangle \tilde{\psi}_{s,j,k}^{[n]} = \tilde{Q}_{n,s,j}(\tilde{P}_{n,j+1}(\tilde{f})).$$

By Lemma 8.6 (iii) we get $\tilde{R}_{n,s,j}(\tilde{P}_{n,j+1}(\tilde{f})) = \tilde{Q}_{n,s,j}\tilde{f}$. □

9 Infinite Direct Sums and the Compactly Supported Interpolating Tensor Product MRA

We assume that $K = \mathbb{R}$ or $K = \mathbb{C}$ throughout this section. The definition of locally convex direct sums from [39] is used here.

Theorem 9.1. *Let $n \in \mathbb{Z}_+$ and $j_0 \in \mathbb{Z}$. Let $E = C_u(\mathbb{R}^n, K)$ or $E = C_0(\mathbb{R}^n, K)$. Suppose that the mother scaling function φ is Lipschitz continuous and let $\varphi^{[n]}$ be the tensor product mother scaling function generated by φ . Then*

$$V_{n,j_0}^E + \sum_{j=j_0}^{\infty} W_{n,j}^E \neq E.$$

Proof. Let

$$A := V_{n,j_0}^E \dot{+} \sum_{j=j_0}^{\infty} W_{n,j}^E.$$

Define function $f \in C_0(\mathbb{R}, K)$ by

$$f(x) := \begin{cases} \sqrt{x}; & x \in [0, 1] \\ 0; & x \leq 0 \text{ or } x \geq 2 \\ -x + 2; & x \in [1, 2] \end{cases}$$

and function $f^{[n]} \in C_0(\mathbb{R}^n, K)$ by

$$f^{[n]} := \bigotimes_{k=1}^n f.$$

Functions f and $f^{[n]}$ are not Lipschitz continuous. As φ is Lipschitz continuous all the functions $P_{n,j}^{(0)}g$ are Lipschitz continuous for each $g \in C_0(\mathbb{R}^n, K)$. A is a locally convex space with an inductive limit topology. It follows from [39, theorem 6.2] that the space A is complete.

Suppose that A would be equal to E as a set. Then we would have $f^{[n]} \in A$. It follows from the definition of the locally convex direct sum that

$$f^{[n]} \in V_{n,j_0}^E \dot{+} \sum_{j=j_0}^{j_1} W_{n,j}^E$$

for some $j_1 \in \mathbb{Z}$, $j_1 \geq j_0$. Now $f^{[n]} = P_{n,j_1+1}^{(0)}f^{[n]}$. It follows that $f^{[n]}$ is Lipschitz continuous, which is a contradiction. Hence A is not equal to E as a set. \square

However, it follows from Equations (58) and (59) that

$$C_0(\mathbb{R}^n, K) =_{\text{n.s.}} \text{clos} \left(\bigcup_{l=j_0}^{\infty} \left(V_{n,j_0}^{(0)} \dot{+} \sum_{j=j_0}^l W_{n,j}^{(0)} \right) \right)$$

and by Lemma 6.13 and Equation (42) we have

$$C_u(\mathbb{R}^n, K) =_{\text{n.s.}} \text{clos} \left(\bigcup_{l=j_0}^{\infty} \left(V_{n,j_0}^{(u)} \dot{+} \sum_{j=j_0}^l W_{n,j}^{(u)} \right) \right).$$

10 Besov Space Norm Equivalence

10.1 Results from Donoho [16]

Donoho has derived norm equivalences for the Besov and Triebel-Lizorkin spaces in the one-dimensional case using the interpolating wavelet expansion (in our notation and norming)

$$f = \sum_{k \in \mathbb{Z}} a_k \varphi_{j_0,k} + \sum_{j \geq j_0} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k}$$

for an arbitrary function $f \in C_0(\mathbb{R})$ [16]. Numbers $R \in \mathbb{R}_+$ and $D \in \mathbb{N}$ are defined so that the mother scaling function φ is Hölder continuous of order R and the collection of formal sums $\sum_k a_k \varphi(t - k)$

contains all the polynomials of degree D . Define $\mathbf{a} = (a_k)_{k \in \mathbb{Z}}$ and $\mathbf{b}_j = (b_{j,k})_{k \in \mathbb{Z}}$ where $j \in \mathbb{Z}$, $j \geq j_0$. The equivalent norm for the Besov space is

$$\|f\| = 2^{-\frac{j_0}{2}} \|\mathbf{a}\|_{l^p} + \left\| \left(2^{(\sigma - \frac{1}{p})j} \|\mathbf{b}_j\|_{l^p} \right)_{j \geq j_0} \right\|_{l^q},$$

where $\min\{R, D\} > \sigma > n/p$ and $p, q \in]0, \infty]$.

10.2 Norm Equivalence for the Besov Spaces in the n -dimensional Case

This derivation is based on the corresponding one-dimensional derivation in [16]. The cases $p < 1$ or $q < 1$ yielding quasi-Banach spaces $B_{p,q}^\sigma$ are not discussed in this article. We assume that $K = \mathbb{R}$ or $K = \mathbb{C}$ and $n \in \mathbb{Z}_+$ throughout this section.

We give first some definitions similar to those in [34] related to orthonormal wavelets.

Definition 10.1. Let $\bar{\varphi} : \mathbb{R}^n \rightarrow K$ be a mother scaling function of an orthonormal wavelet family. When $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, and $f \in L^\infty(\mathbb{R}^n, K)$ define

$$\bar{\alpha}_{j,\mathbf{k}}(f) := 2^{nj} \int_{\mathbf{x} \in \mathbb{R}^n} \bar{\varphi}^*(2^j \mathbf{x} - \mathbf{k}) f(\mathbf{x}) d\tau.$$

The spaces $V_j(p)$ that are defined in [34] are denoted by $\bar{V}_j(p)$ in this document. Space $\bar{V}_j(p)$ is a closed subspace of $L^p(\mathbb{R}^n, K)$ for each $j \in \mathbb{Z}$ and $p \in [1, \infty]$ at least when the mother scaling function $\bar{\varphi}$ is continuous and compactly supported.

Definition 10.2. Let $\bar{\varphi} : \mathbb{R}^n \rightarrow K$ be a scaling function of an orthonormal wavelet family. Let $p \in [1, \infty]$ and $j \in \mathbb{Z}$. Define operator $\bar{P}_j^{(p)} : L^p(\mathbb{R}^n, K) \rightarrow \bar{V}_j(p)$ by

$$(\bar{P}_j^{(p)} f)(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \bar{\alpha}_{j,\mathbf{k}}(f) \bar{\varphi}(2^j \mathbf{x} - \mathbf{k})$$

for all $f \in L^p(\mathbb{R}^n, K)$ and $\mathbf{x} \in \mathbb{R}^n$. Define also $\bar{Q}_j^{(p)} := \bar{P}_{j+1}^{(p)} - \bar{P}_j^{(p)}$.

When $\bar{\varphi}$ is a compactly supported and continuous function operator $\bar{P}_j^{(p)}$ is a continuous linear projection of $L^p(\mathbb{R}^n, K)$ onto $\bar{V}_j(p)$ for each $j \in \mathbb{Z}$ and $p \in [1, \infty]$.

Definition 10.3. When $m \in \mathbb{N}$ define

$$I_{\Sigma=}^{[n]}(m) := \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 = m\}.$$

Definition 10.4. When $m \in \mathbb{N}$, $r \in [0, 1]$, and $f \in C^m(\mathbb{R}^n)$ define

$$H(f; m, r) := \max_{\alpha \in I_{\Sigma=}^{[n]}(m)} \sup \left\{ \frac{|(D^\alpha f)(\mathbf{x}) - (D^\alpha f)(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|^r} : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \wedge \mathbf{x} \neq \mathbf{y} \right\}.$$

Definition 10.5. Let $j \in \mathbb{Z}$. Define

$$S_j f := \left(f \left(\frac{\mathbf{k}}{2^j} \right) \right)_{\mathbf{k} \in \mathbb{Z}^n}$$

for all functions $f : \mathbb{R}^n \rightarrow K$.

Definition 10.6. Let $j \in \mathbb{Z}$ and $\mathbf{b} \in \mathbb{R}^n$. Define

$$S_{j,\mathbf{b}}f := \left(f \left(\frac{\mathbf{k}}{2^j} + \mathbf{b} \right) \right)_{\mathbf{k} \in \mathbb{Z}^n}$$

for all functions $f : \mathbb{R}^n \rightarrow K$.

Norm $\|\cdot\|_{B_{p,q}^\sigma(\mathbb{R}^n);j_0}^{(o)}$ is an equivalent norm for the Besov space $B_{p,q}^\sigma(\mathbb{R}^n)$ and characterizes $B_{p,q}^\sigma(\mathbb{R}^n)$ on $L^p(\mathbb{R}^n)$. This is proved in [34, chapter 2.9 proposition 4].

Definition 10.7. Let $p \in [1, \infty]$, $q \in [1, \infty]$, and $\sigma \in \mathbb{R}_+$. Let $j_0 \in \mathbb{Z}$. Let $P_{n,j}^{(u)}$ and $Q_{n,j}^{(u)}$, $j \in \mathbb{Z}$, be the projection operators belonging to a compactly supported interpolating tensor product MRA of $C_u(\mathbb{R}^n)$. When $f \in C_u(\mathbb{R}^n, K) \cap B_{p,q}^\sigma(\mathbb{R}^n)$ define

$$\|f\|_{B_{p,q}^\sigma(\mathbb{R}^n);j_0}^{(i)} := \left\| P_{n,j_0}^{(u)} f |L^p(\mathbb{R}^n) \right\| + \|\mathbf{h}\|_{l^q(\mathbb{N} + j_0)}$$

where

$$h_j := 2^{j\sigma} \left\| Q_{n,j}^{(u)} f |L^p(\mathbb{R}^n) \right\|, \quad j \in \mathbb{N} + j_0$$

$$\mathbf{h} := (h_j)_{j=j_0}^\infty.$$

Definition 10.8. Let $p \in [1, \infty]$, $q \in [1, \infty]$, and $\sigma \in \mathbb{R}_+$. Let $j_0 \in \mathbb{Z}$. Let $\tilde{\varphi}_{j,\mathbf{k}}^{[n]}$ and $\tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]}$, $j \in \mathbb{Z}$, $\mathbf{s} \in J_+(n)$, $\mathbf{k} \in \mathbb{Z}^n$ be the dual scaling functions and dual wavelets belonging to a compactly supported interpolating tensor product MRA of $C_u(\mathbb{R}^n, K)$. When $f \in C_u(\mathbb{R}^n, K) \cap B_{p,q}^\sigma(\mathbb{R}^n)$ define

$$\|f\|_{B_{p,q}^\sigma(\mathbb{R}^n);j_0}^{(w)} := \left\| \left(\left\langle \tilde{\varphi}_{j_0,\mathbf{k}}^{[n]}, f \right\rangle \right)_{\mathbf{k} \in \mathbb{Z}^n} \right\|_p + \left\| \left(2^{(\sigma - \frac{n}{p})j} \left\| \left(\left\langle \tilde{\psi}_{\mathbf{s},j,\ell}^{[n]}, f \right\rangle \right)_{\mathbf{s} \in J_+(n), \ell \in \mathbb{Z}^n} \right\|_p \right)_{j=j_0}^\infty \right\|_q.$$

Definition 10.9. When $j \in \mathbb{Z}$ and $p \in [1, \infty]$ define

$$V_{n,j}^{(u)}(p) := \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \varphi_{j,\mathbf{k}}^{[n]}(\mathbf{x}) : \mathbf{a} \in l^p(\mathbb{Z}^n) \right\}$$

and $\|f|V_{n,j}^{(u)}(p)\| := \|f|L^p(\mathbb{R}^n)\|$ for all $f \in V_{n,j}^{(u)}(p)$.

Definition 10.10. Let $j \in \mathbb{Z}$ and $p \in [1, \infty]$. When $T \in \mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ define

$$\|T\|_{V_{n,j}^{(u)}(p)} := \left\| (f|_{V_{n,j}^{(u)}(p)}) | \mathcal{L}(V_{n,j}^{(u)}(p), L^p(\mathbb{R}^n)) \right\|.$$

Definition 10.11. When $j \in \mathbb{Z}$ and $p \in [1, \infty]$ define

$$W_{n,j}^{(u)}(p) := \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{s} \in J_+(n)} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{s}, \mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}(\mathbf{x}) : \mathbf{a} \in l^p(J_+(n) \times \mathbb{Z}^n) \right\}$$

and $\|f|W_{n,j}^{(u)}(p)\| := \|f|L^p(\mathbb{R}^n)\|$ for all $f \in W_{n,j}^{(u)}(p)$. When $j \in \mathbb{Z}$, $\mathbf{s} \in \{0, 1\}^n$, and $p \in [1, \infty]$ define

$$W_{n,\mathbf{s},j}^{(u)}(p) := \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}(\mathbf{x}) : \mathbf{a} \in l^p(\mathbb{Z}^n) \right\}$$

and $\|f|W_{n,\mathbf{s},j}^{(u)}(p)\| := \|f|L^p(\mathbb{R}^n)\|$ for all $f \in W_{n,\mathbf{s},j}^{(u)}(p)$.

Definition 10.12. When $p \in [1, \infty]$ define

$$c_{\text{interp}}(p) := \begin{cases} \frac{p}{p-1}; & p \in]1, \infty[\\ 1; & p = 1 \vee p = \infty. \end{cases}$$

Lemma 10.13. Let $\bar{\varphi}$ be a mother scaling function of a 0-regular orthonormal MRA of $L^2(\mathbb{R}^n)$. Then

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} \bar{\varphi}(\mathbf{x} - \mathbf{k}) = \left(\int_{\mathbf{y} \in \mathbb{R}^n} \bar{\varphi}^*(\mathbf{y}) d\tau \right)^{-1}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Definition 10.14. Let $\bar{\varphi}$ be a mother scaling function of a 0-regular orthonormal MRA of $L^2(\mathbb{R}^n)$. Define

$$\nu_0 := \int_{\mathbf{y} \in \mathbb{R}^n} \bar{\varphi}^*(\mathbf{y}) d\tau.$$

Lemma 10.15. Let $\bar{\varphi}$ be a mother scaling function of a 0-regular orthonormal MRA of $L^2(\mathbb{R}^n)$. Let $f \in C_b(\mathbb{R}^n)$. Suppose that $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$. Then

$$\left| \bar{\alpha}_{j,\mathbf{k}}(f) - \nu_0 f\left(\frac{\mathbf{k}}{2^j}\right) \right| \leq \|\bar{\varphi}\|_1 \omega(f; 2^{-j} r_{\text{supp}}(\bar{\varphi}))$$

Lemma 10.16. Suppose that $\bar{\varphi} : \mathbb{R}^n \rightarrow K$ is a compactly supported and continuous mother scaling function of a 0-regular orthonormal wavelet family. Then

$$\left\| f - \bar{P}_j^{(\infty)} f \right\|_{\infty} \leq (|\nu_0| + \|\bar{\varphi}\|_1) \|\bar{\varphi}\|_{\infty} N_{\text{cover}}(\bar{\varphi}) \omega(f; r_{\text{supp}}(\bar{\varphi}) \cdot 2^{-j})$$

for all $j \in \mathbb{Z}$ and $f \in C_b(\mathbb{R}^n, K)$.

Proof. Use Lemmas 10.13 and 10.15. □

See also Jackson's inequality [42, proposition 9.6].

Lemma 10.17. Suppose that $q \in [1, \infty]$ and $t \in \mathbb{R}_+$. Let $a(j, j') := 2^{-|j'-j|t}$ for all $j, j' \in \mathbb{N}$. Define

$$A\mathbf{b} := \left(\sum_{j'=0}^{\infty} a(j, j') b_{j'} \right)_{j \in \mathbb{N}}$$

for all $\mathbf{b} \in l^q(\mathbb{N}, K)$. Then $A \in \mathcal{L}(l^q(\mathbb{N}, K), l^q(\mathbb{N}, K))$.

Proof. Cases $q = 1$ and $q = \infty$ can be proved by starting from the expression of $A\mathbf{b}$ where $\mathbf{b} \in l^q(\mathbb{N}, K)$. When $q \in]1, \infty[$ the proof is based on interpolation of Banach spaces, see [3]. Suppose first that $K = \mathbb{C}$. Let ν be the counting measure on \mathbb{N} . Operator A is admissible with respect to couple (l^1, l^{∞}) . As $l^p = L^p(\mathbb{N}, \nu)$ for all $p \in]1, \infty[$ it follows by [3, corollary IV.1.8] that A is a continuous linear operator from l^q into l^q . If $K = \mathbb{R}$ we have $l^q(\mathbb{N}, \mathbb{R}) \subset_{\text{c.s.}} l^q(\mathbb{N}, \mathbb{C})$ and $A[l^q(\mathbb{N}, \mathbb{R})] \subset l^q(\mathbb{N}, \mathbb{R})$ so the lemma is true in this case, too. □

Lemma 10.18. Let

$$A_j f := (\bar{\alpha}_{j,\mathbf{k}}(f))_{\mathbf{k} \in \mathbb{Z}^n}$$

for all $f \in L^1(\mathbb{R}^n) \cup L^{\infty}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$. Then $A_j f \in l^p(\mathbb{Z}^n)$ for all $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, and $j \in \mathbb{Z}$. Furthermore, there exists constant $c_1 \in \mathbb{R}_+$ so that

$$\|A_j|_{\mathcal{L}(L^p(\mathbb{R}^n), l^p(\mathbb{Z}^n))}\| \leq 2^{\frac{nj}{p}} c_1$$

for all $p \in [1, \infty]$ and $j \in \mathbb{Z}$.

Lemma 10.19. Let $p \in [1, \infty]$, $v \in C_{\text{com}}(\mathbb{R}^n)$, and $\mathbf{b} \in \mathbb{R}^n$. There exists constant $c_1 \in \mathbb{R}_+$ so that for all $j \in \mathbb{Z}$ and for all functions

$$f(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] v(2^j \mathbf{x} - \mathbf{k})$$

where $\mathbf{a} \in \mathbb{C}^{\mathbb{Z}^n}$ we have

$$\|S_{j,\mathbf{b}} f\|_p \leq 2^{\frac{nj}{p}} c_1 \|f\|_p.$$

Lemma 10.20. Suppose that $v \in C_{\text{com}}(\mathbb{R}^n)$ and $v(\mathbf{k}) = \delta_{\mathbf{k},0}$ for all $\mathbf{k} \in \mathbb{Z}^n$. Let $j \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{C}^{\mathbb{Z}^n}$. Let

$$f(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] v(2^j \mathbf{x} - \mathbf{k}) \quad (66)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and assume that series (66) converges absolutely for each $\mathbf{x} \in \mathbb{R}^n$. Furthermore, assume that $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$. Then $\mathbf{a} \in l^p(\mathbb{Z}^n)$.

Proof. Use Lemma 10.19. □

Lemma 10.21. Suppose that $v_k \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for each $k \in \mathbb{N}$. Define

$$\iota_p(\mathbf{a}) := \mathbf{x} \mapsto \sum_{k \in \mathbb{N}} \mathbf{a}[k] v_k(\mathbf{x}) \quad (67)$$

for all $\mathbf{a} \in l^p$ and $p \in [1, \infty]$. When $p = 1$ or $p = \infty$ assume that the series in (67) converges absolutely for all $\mathbf{a} \in l^p$ and $\mathbf{x} \in \mathbb{R}^n$. Then ι_p is an operator from l^p into $L^p(\mathbb{R}^n)$ for each $p \in]1, \infty[$.

Lemma 10.22. Assume that the following conditions are true.

(A1) I is a countably infinite set.

(A2) $\forall \alpha \in I, j \in \mathbb{Z} : v_{j,\alpha} \in C_{\text{com}}(\mathbb{R}^n)$

(A3) Either of the following conditions is true:

(A3.1) We have $m(\alpha) \in \mathbb{Z}_+$, $r_{j,\alpha,k} \in \mathbb{C}$, and $\mathbf{s}_{j,\alpha,k} \in \mathbb{R}^n$ for all $\alpha \in I$, $j \in \mathbb{Z}$, and $k \in Z(m(\alpha))$. Furthermore,

$$\tilde{v}_{j,\alpha} = \sum_{k=1}^{m(\alpha)} r_{j,\alpha,k} \delta(\cdot - \mathbf{s}_{j,\alpha,k})$$

for all $\alpha \in I$ and $j \in \mathbb{Z}$.

(A3.2) We have $\tilde{v}_{j,\alpha} \in C_{\text{com}}(\mathbb{R}^n)$ for all $\alpha \in I$ and $j \in \mathbb{Z}$. The set $\{\beta \in I : \text{supp } \tilde{v}_{j,\alpha} \cap \text{supp } v_{j,\beta}\}$ is finite for each $\alpha \in I$ and $j \in \mathbb{Z}$.

(A4) $\forall j \in \mathbb{Z}, \alpha \in I, \beta \in I : \langle \tilde{v}_{j,\alpha}, v_{j,\beta} \rangle = \delta_{\alpha,\beta}$

(A5) The series

$$\sum_{\alpha \in I} \mathbf{a}[\alpha] v_{j,\alpha}(\mathbf{x})$$

converges absolutely for each $\mathbf{a} \in l^\infty(I)$, $j \in \mathbb{Z}$, and $\mathbf{x} \in \mathbb{R}^n$.

Define

$$(\iota_{p,j}(\mathbf{a}))(\mathbf{x}) := \sum_{\alpha \in I} \mathbf{a}[\alpha] v_{j,\alpha}(\mathbf{x})$$

for all $\mathbf{a} \in l^p(I)$, $j \in \mathbb{Z}$, $\mathbf{x} \in \mathbb{R}^n$, and $p \in \{1, \infty\}$. Define also $A_{p,j} := \iota_{p,j}[l^p(I)]$ for all $p \in \{1, \infty\}$, $j \in \mathbb{Z}$ and $\|f|_{A_{p,j}}\| := \|f\|_p$ for all $p \in \{1, \infty\}$, $j \in \mathbb{Z}$, and $f \in A_{p,j}$. Assume further that the following conditions are true.

(A6) $\iota_{1,j}$ is a topological isomorphism from $l^1(I)$ onto $A_{1,j}$ for each $j \in \mathbb{Z}$.

(A7) $\iota_{\infty,j}$ is a topological isomorphism from $l^\infty(I)$ onto $A_{\infty,j}$ for each $j \in \mathbb{Z}$.

(A8) $A_{1,j} \subset_{c.s.} L^1(\mathbb{R}^n)$ and $A_{\infty,j} \subset_{c.s.} L^\infty(\mathbb{R}^n)$ for each $j \in \mathbb{Z}$.

(A9) $\exists_{\text{def}} c_1 \in \mathbb{R}_+ : \forall j \in \mathbb{Z} : \|\iota_{1,j}\| \leq 2^{-nj} c_1$

(A10) $\exists_{\text{def}} c_2 \in \mathbb{R}_+ : \forall j \in \mathbb{Z} : \|(\iota_{1,j})^{-1}\| \leq 2^{nj} c_2$

(A11) $\exists_{\text{def}} c_3 \in \mathbb{R}_+ : \forall j \in \mathbb{Z} : \|\iota_{\infty,j}\| \leq c_3$

(A12) $\exists_{\text{def}} c_4 \in \mathbb{R}_+ : \forall j \in \mathbb{Z} : \|(\iota_{\infty,j})^{-1}\| \leq c_4$

Then

(i) Set

$$(\iota_{p,j}(\mathbf{a}))(\mathbf{x}) := \sum_{\alpha \in I} \mathbf{a}[\alpha] v_{j,\alpha}(\mathbf{x})$$

for all $\mathbf{a} \in l^p(I)$, $j \in \mathbb{Z}$, $\mathbf{x} \in \mathbb{R}^n$, and $p \in]1, \infty[$. Then $\iota_{p,j}$ is an operator from $l^p(I)$ into $L^p(\mathbb{R}^n)$.

(ii) Define $A_{p,j} := \iota_{p,j}[l^p(I)]$ for all $p \in]1, \infty[$ and $j \in \mathbb{Z}$. Now

$$A_{p,j} =_{\text{tvs}} (A_{1,j}, A_{\infty,j})_{1-\frac{1}{p}, p}$$

for all $p \in]1, \infty[$ and $j \in \mathbb{Z}$.

(iii) Function $\iota_{p,j}$ is a topological isomorphism from $l^p(I)$ onto $A_{p,j}$ for each $p \in [1, \infty]$ and $j \in \mathbb{Z}$.

(iv) $\exists c_5 \in \mathbb{R}_+ : \forall p \in]1, \infty], j \in \mathbb{Z} : \|\iota_{p,j}\| \leq \frac{p}{p-1} 2^{-\frac{nj}{p}} c_5$

(v) $\exists c_6 \in \mathbb{R}_+ : \forall p \in]1, \infty], j \in \mathbb{Z} : \|(\iota_{p,j})^{-1}\| \leq \frac{p}{p-1} 2^{\frac{nj}{p}} c_6$.

Proof.

(i) This is a consequence of Lemma 10.21.

(ii) Let $p_1 \in]1, \infty[$ and $j_1 \in \mathbb{Z}$. Let

$$\eta := (\iota_{1,j_1}, \iota_{\infty,j_1})_{1-\frac{1}{p_1}, p_1}$$

Suppose first that $f \in A_{p_1,j_1}$. Now $f = \iota_{p_1,j_1}(\mathbf{a})$ for some $\mathbf{a} \in l^{p_1}(I)$. As

$$l^{p_1} =_{\text{tvs}} (l^1, l^\infty)_{1-\frac{1}{p_1}, p_1}$$

we have $\mathbf{a} = \mathbf{b} + \mathbf{c}$ for some $\mathbf{b} \in l^1(I)$ and $\mathbf{c} \in l^\infty(I)$. It follows that

$$f = \iota_{1,j_1}(\mathbf{b}) + \iota_{\infty,j_1}(\mathbf{c}) = \eta(\mathbf{a}) \in (A_{1,j_1}, A_{\infty,j_1})_{1-\frac{1}{p_1}, p_1}.$$

Suppose then that

$$g \in (A_{1,j_1}, A_{\infty,j_1})_{1-\frac{1}{p_1}, p_1}.$$

Now $g = w + z$ for some $w \in A_{1,j_1}$ and $z \in A_{\infty,j_1}$. Furthermore, $w = \iota_{1,j_1}(\mathbf{r})$ for some $\mathbf{r} \in l^1(I)$ and $z = \iota_{\infty,j_1}(\mathbf{s})$ for some $\mathbf{s} \in l^\infty(I)$. We have

$$g(\mathbf{x}) = \sum_{\alpha \in I} (\mathbf{r}[\alpha] + \mathbf{s}[\alpha]) v_{j_1,\alpha}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and it follows that $g = \eta(\mathbf{r} + \mathbf{s})$. Hence

$$\eta^{-1}(g) = \mathbf{r} + \mathbf{s} \in (l^1(I), l^\infty(I))_{1-\frac{1}{p_1}, p_1}$$

from which it follows that $\mathbf{r} + \mathbf{s} \in l^p(I)$. Thus $g = \iota_{p_1, j_1}(\mathbf{r} + \mathbf{s}) \in A_{p_1, j_1}$.

- (iii) Let $\mathbf{b} \in l^p(I) \setminus \{0\}$. There exists $\alpha_1 \in I$ so that $\mathbf{b}[\alpha_1] \neq 0$. If (A3.1) is true we have $\langle \tilde{v}_{j_1, \alpha_1}, \iota_{p_1, j_1}(\mathbf{b}) \rangle = \mathbf{b}[\alpha_1] \neq 0$. If (A3.2) is true we have

$$J := \{\beta \in I : \text{supp } \tilde{v}_{j_1, \alpha_1} \cap \text{supp } v_{j_1, \beta} \neq \{0\}\}$$

and it follows that

$$\begin{aligned} \langle \tilde{v}_{j_1, \alpha_1}, \iota_{p_1, j_1}(\mathbf{b}) \rangle &= \int_{\mathbf{x} \in \mathbb{R}^n} \tilde{v}_{j_1, \alpha_1}(\mathbf{x}) \left(\sum_{\beta \in J} \mathbf{b}[\beta] v_{j_1, \beta}(\mathbf{x}) \right) d\tau = \sum_{\beta \in J} \mathbf{b}[\beta] \langle \tilde{v}_{j_1, \alpha_1}, v_{j_1, \beta} \rangle \\ &= \mathbf{b}[\alpha_1] \neq 0. \end{aligned}$$

Consequently $\iota_{p_1, j_1}(\mathbf{b}) \neq 0$. Hence ι_{p_1, j_1} is an injection.

By Lemma 10.21 function ι_{p_1, j_1} is continuous. By the Inverse Mapping Theorem function ι_{p_1, j_1}^{-1} is continuous. Thus ι_{p_1, j_1} is a topological isomorphism from Banach space $l^{p_1}(I)$ onto Banach space A_{p_1, j_1} .

- (iv) Let $p_1 \in]1, \infty[$ and $j_1 \in \mathbb{Z}$. Now by Lemma 3.23

$$\|\iota_{p_1, j_1}\| \leq \frac{p}{p-1} \|\iota_{1, j_1}\|^{\frac{1}{p_1}} \|\iota_{\infty, j_1}\|^{1-\frac{1}{p_1}} \leq c_5 \frac{p}{p-1} \cdot 2^{-\frac{nj_1}{p_1}}$$

where $c_5 := c_1^{\frac{1}{p_1}} c_3^{1-\frac{1}{p_1}}$.

- (v) Let $p_1 \in]1, \infty[$ and $j_1 \in \mathbb{Z}$. Now by Lemma 3.23

$$\|(\iota_{p_1, j_1})^{-1}\| \leq \frac{p}{p-1} \|(\iota_{1, j_1})^{-1}\|^{\frac{1}{p_1}} \|(\iota_{\infty, j_1})^{-1}\|^{1-\frac{1}{p_1}} \leq c_6 \frac{p}{p-1} \cdot 2^{\frac{nj_1}{p_1}}$$

where $c_6 := c_2^{\frac{1}{p_1}} c_4^{1-\frac{1}{p_1}}$.

□

Lemma 10.23. Let $f \in C_{\text{com}}(\mathbb{R}^n)$, $\mathbf{b} \in \mathbb{R}^n$ and suppose that $f(\mathbf{k} + \mathbf{b}) = \delta_{\mathbf{k}, 0}$ for all $\mathbf{k} \in \mathbb{Z}^n$. Define

$$(\iota_j(\mathbf{a}))(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] f(2^j \mathbf{x} - \mathbf{k})$$

for all $\mathbf{a} \in l^1(\mathbb{Z}^n)$ and $j \in \mathbb{Z}$. Define also

$$A_j := \{\iota_j(\mathbf{a}) : \mathbf{a} \in l^1(\mathbb{Z}^n)\}$$

for all $j \in \mathbb{Z}$ and $\|g|_{A_j}\| := \|g|_{L^1(\mathbb{R}^n)}\|$ for all $g \in A_j$ and $j \in \mathbb{Z}$. Then

- (i) ι_j is a topological isomorphism from $l^1(\mathbb{Z}^n)$ onto A_j for each $j \in \mathbb{Z}$.
- (ii) $\exists c_1 \in \mathbb{R}_+ : \forall j \in \mathbb{Z} : \|\iota_j\| \leq c_1 \cdot 2^{-nj}$

$$(iii) \exists c_2 \in \mathbb{R}_+ : \forall j \in \mathbb{Z} : \left\| (\iota_j)^{-1} \right\| \leq c_2 \cdot 2^{nj}$$

Proof. Use Lemmas 3.19 and 10.19. \square

Lemma 10.24. *Let*

$$(\iota_{p,\mathbf{s},j}(\mathbf{a}))(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $p \in [1, \infty]$, $\mathbf{s} \in \{0, 1\}^n$, $j \in \mathbb{Z}$, $\mathbf{a} \in l^p(\mathbb{Z}^n)$, and $\mathbf{x} \in \mathbb{R}^n$. Then

(i) We have

$$W_{n,\mathbf{s},j}^{(u)}(p) =_{\text{tvs}} \left(W_{n,\mathbf{s},j}^{(u)}(1), W_{n,\mathbf{s},j}^{(u)}(\infty) \right)$$

for all $p \in]1, \infty[$, $\mathbf{s} \in \{0, 1\}^n$, and $j \in \mathbb{Z}$.

(ii) Function $\iota_{p,\mathbf{s},j}$ is a topological isomorphism from $l^p(\mathbb{Z}^n)$ onto $W_{n,\mathbf{s},j}^{(u)}(p)$ for all $p \in]1, \infty[$, $\mathbf{s} \in \{0, 1\}^n$, and $j \in \mathbb{Z}$.

$$(iii) \exists c_1 \in \mathbb{R}_+ : \forall p \in [1, \infty], j \in \mathbb{Z}, \mathbf{s} \in \{0, 1\}^n : \|\iota_{p,\mathbf{s},j}\| \leq c_1 \cdot c_{\text{interp}}(p) \cdot 2^{-\frac{nj}{p}}$$

$$(iv) \exists c_2 \in \mathbb{R}_+ : \forall p \in [1, \infty], j \in \mathbb{Z}, \mathbf{s} \in \{0, 1\}^n : \left\| (\iota_{p,\mathbf{s},j})^{-1} \right\| \leq c_2 \cdot c_{\text{interp}}(p) \cdot 2^{\frac{nj}{p}}$$

Proof. Use Lemmas 3.10, 3.20, 10.22, and 10.23. \square

Lemma 10.25. *Let $\bar{V}_j(p)$ belong to an orthonormal MRA on \mathbb{R}^n . Define functions $\iota_{p,j}$ by*

$$(\iota_{p,j}(\mathbf{a}))(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] \bar{\varphi}_{j,\mathbf{k}}(\mathbf{x})$$

for all $\mathbf{a} \in l^p(\mathbb{Z}^n)$, $\mathbf{x} \in \mathbb{R}^n$, $p \in [1, \infty]$, and $j \in \mathbb{Z}$. Then

$$(i) \forall j \in \mathbb{Z}, p \in]1, \infty[: \bar{V}_j(p) =_{\text{tvs}} (\bar{V}_j(1), \bar{V}_j(\infty))_{1-\frac{1}{p}, p}$$

(ii) Function $\iota_{p,j}$ is a topological isomorphism from $l^p(\mathbb{Z}^n)$ onto $\bar{V}_j(p)$ for each $j \in \mathbb{Z}$ and $p \in [1, \infty]$.

$$(iii) \exists c_1 \in \mathbb{R}_+ : \forall j \in \mathbb{Z}, p \in [1, \infty] : \|\iota_{p,j}\| \leq c_1 \cdot c_{\text{interp}}(p) \cdot 2^{-\frac{nj}{p}}$$

$$(iv) \exists c_2 \in \mathbb{R}_+ : \forall j \in \mathbb{Z}, p \in [1, \infty] : \left\| (\iota_{p,j})^{-1} \right\| \leq c_2 \cdot c_{\text{interp}}(p) \cdot 2^{\frac{nj}{p}}$$

Proof. Use Lemmas 3.19, 3.20, and 10.22. \square

Lemma 10.26. *Define functions $\iota_{p,j}$ by*

$$(\iota_{p,j}(\mathbf{a}))(\mathbf{x}) := \sum_{\mathbf{s} \in J_+(n)} \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{s}, \mathbf{k}] \psi_{\mathbf{s},j,\mathbf{k}}^{[n]}(\mathbf{x})$$

for all $\mathbf{a} \in l^p(J_+(n) \times \mathbb{Z}^n)$, $\mathbf{x} \in \mathbb{R}^n$, $p \in [1, \infty]$, and $j \in \mathbb{Z}$. Then

$$(i) \forall j \in \mathbb{Z}, p \in]1, \infty[: \left(W_{n,j}^{(u)}(1), W_{n,j}^{(u)}(\infty) \right)_{1-\frac{1}{p}, p} =_{\text{tvs}} W_{n,j}^{(u)}(p) \subset_{c.s.} L^p(\mathbb{R}^n)$$

(ii) Function $\iota_{p,j}$ is a topological isomorphism from $l^p(J_+(n) \times \mathbb{Z}^n)$ onto $W_{n,j}^{(u)}(p)$ for each $j \in \mathbb{Z}$ and $p \in [1, \infty]$.

$$(iii) \exists c_1 \in \mathbb{R}_+ : \forall j \in \mathbb{Z}, p \in [1, \infty] : \|\iota_{p,j}\| \leq c_1 \cdot c_{\text{interp}}(p) \cdot 2^{-\frac{nj}{p}}$$

$$(iv) \exists c_2 \in \mathbb{R}_+ : \forall j \in \mathbb{Z}, p \in [1, \infty] : \left\| (\iota_{p,j})^{-1} \right\| \leq c_2 \cdot c_{\text{interp}}(p) \cdot 2^{\frac{nj}{p}}$$

Proof. Use Lemmas 6.19, 10.19, and 10.22 and Equation (28). \square

Lemma 10.27. *Let $p \in [1, \infty]$. Then*

$$\exists c_1 \in \mathbb{R}_+ : \forall j \in \mathbb{Z}, j' \in \mathbb{Z} : j' \geq j \implies \left\| P_{n,j}^{(u)} \right\|_{\bar{V}_p(j')} \leq c_1 \cdot 2^{n(j'-j)/p}.$$

Proof. Use Lemmas 3.19, 3.23, 10.19, and 10.25. \square

We have [34]

$$\forall p \in [1, \infty] : \exists c_1 \in \mathbb{R}_+ : \forall j \in \mathbb{Z} : \left\| \bar{P}_j^{(p)} \right\| \leq c_1. \quad (68)$$

Lemma 10.28. *Let $j \in \mathbb{Z}$, $s \in \mathbb{R}_+$, $p \in [1, \infty]$, and $q \in [1, \infty]$. Then $P_{n,j}^{(u)} f \in V_{n,j}^{(u)}(p)$ for all $f \in B_{p,q}^s(\mathbb{R}^n)$.*

Proof. We have

$$f = \bar{P}_j^{(p)} f + \sum_{j'=j}^{\infty} \bar{Q}_{j'}^{(p)} f = \bar{P}_j^{(\infty)} f + \sum_{j'=j}^{\infty} \bar{Q}_{j'}^{(\infty)} f$$

and

$$\left\| P_{n,j}^{(u)} f \right\|_p \leq \left\| P_{n,j}^{(u)} \bar{P}_j^{(p)} f \right\|_p + \sum_{j'=j}^{\infty} \left\| P_{n,j}^{(u)} \bar{Q}_{j'}^{(p)} f \right\|_p \in \mathbb{R}_0.$$

Consequently $P_{n,j}^{(u)} f \in L^p(\mathbb{R}^n)$. By Lemma 10.20 we have

$$\left(f \left(\frac{\mathbf{k}}{2^j} \right) \right)_{\mathbf{k} \in \mathbb{Z}^n} \in l^p(\mathbb{Z}^n).$$

Thus $P_{n,j}^{(u)} f \in V_{n,j}^{(u)}(p)$. \square

Lemma 10.29. *Let $m \in \mathbb{N}$, $a \in [0, 1]$, and $v \in C_{\text{com}}(\mathbb{R}^n)$. Then there exists $c_1 \in \mathbb{R}_+$ so that for all $\mathbf{b} \in l^\infty(\mathbb{Z}^n)$, $j \in \mathbb{Z}$, and*

$$f = \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{b}[\mathbf{k}] v(2^j \mathbf{x} - \mathbf{k})$$

we have

$$H(f; m, a) \leq c_1 \cdot 2^{j(m+a)} H(v; m, a) \|\mathbf{b}\|_\infty.$$

Lemma 10.30. *Let $s \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, $s > 1$, and $f \in C^s(\mathbb{R}^n)$. Let $m := \lfloor s \rfloor$ and $a := s - m$. Let $\mathbf{x}_0 \in \mathbb{R}^n$, p_m be the m th degree Taylor polynomial at point \mathbf{x}_0 , and $r_m := f - p_m$. Then*

$$|r_m(\mathbf{x})| \leq \frac{n}{(m-1)!} H(f; m, a) \|\mathbf{x} - \mathbf{x}_0\|^s$$

for all $\mathbf{x} \in \mathbb{R}^n$.

See also [21, chapter 10] and [3, chapter V.5].

Lemma 10.31. *Assume that the following conditions are true.*

(B1) $s \in \mathbb{R}_+$ and $v \in C_{\text{com}}(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$

(B2) We have $\tilde{w} \in C_u(\mathbb{R}^n)^*$ and $w \in C_{\text{com}}(\mathbb{R}^n)$.

(B3) Either of the following conditions is true:

(B3.1) We have

$$\tilde{w} = \sum_{k=1}^m r_k \delta(\cdot - \tilde{s}_k)$$

where $m \in \mathbb{Z}_+$ and $r_k \in \mathbb{C}$.

(B3.2) $\tilde{w} \in C_{\text{com}}(\mathbb{R}^n)$.

(B4) We have

$$\begin{aligned} \text{supp } v &\subset \overline{B}_{\mathbb{R}^n}(0; \rho_1), & \rho_1 &\in \mathbb{R}_+, \\ \text{supp } w &\subset \overline{B}_{\mathbb{R}^n}(0; \rho_2), & \rho_2 &\in \mathbb{R}_+, \\ \text{supp } \tilde{w} &\subset \overline{B}_{\mathbb{R}^n}(0; \rho_3), & \rho_3 &\in \mathbb{R}_+. \end{aligned}$$

(B5) We have

$$\begin{aligned} v_{j,\mathbf{k}} &= v(2^j \cdot -\mathbf{k}) \\ w_{j,\mathbf{k}} &= w(2^j \cdot -\mathbf{k}) \\ \tilde{w}_{j,\mathbf{k}} &= 2^{nj} \tilde{w}(2^j \cdot -\mathbf{k}) \end{aligned}$$

where $\forall j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$.

(B6) $\forall \mathbf{k} \in \mathbb{Z}^n, \ell \in \mathbb{Z}^n, j \in \mathbb{Z} : \langle \tilde{w}_{j,\mathbf{k}}, w_{j,\ell} \rangle = \delta_{\mathbf{k},\ell}$.

(B7) Define $M_{j,\mathbf{k}} := \overline{B}_{\mathbb{R}^n}(2^{-j}\mathbf{k}; 2^{-j}\rho_3)$ for all $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^n$. We have $\tilde{z}_{j,\mathbf{k}} \in C(M_{j,\mathbf{k}})^*$ for all $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, and

$$\langle \tilde{w}_{j,\mathbf{k}}, f \rangle = \langle \tilde{z}_{j,\mathbf{k}}, f|_{M_{j,\mathbf{k}}} \rangle$$

for all $j \in \mathbb{Z}$, $\mathbf{k} \in \mathbb{Z}^n$, and $f \in C_u(\mathbb{R}^n)$.

(B8) When $j \in \mathbb{Z}$ define linear function P_j by

$$(P_j f)(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \tilde{z}_{j,\mathbf{k}}, f|_{M_{j,\mathbf{k}}} \rangle w_{j,\mathbf{k}}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$, $j \in \mathbb{Z}$, and $f \in C(\mathbb{R}^n)$.

(B9) Define

$$\begin{aligned} A_{p,j} &:= \left\{ \mathbf{x} \in \mathbb{R}^n \mapsto \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{a}[\mathbf{k}] v_{j,\mathbf{k}}(\mathbf{x}) : \mathbf{a} \in l^p(\mathbb{Z}^n) \right\} \\ \|f|_{A_{p,j}}\| &:= \|f\|_p \end{aligned}$$

for all $p \in [1, \infty]$ and $j \in \mathbb{Z}$.

(B10) Pair (\tilde{w}, w) spans all the polynomials of n variables and of degree at most $\lceil s \rceil - 1$.

Then

$$\begin{aligned} &\exists c_1 \in \mathbb{R}_+ : \forall p \in [1, \infty], j \in \mathbb{Z}, j' \in \mathbb{Z} : \\ &j' \leq j \implies \sup \left\{ \|(I - P_j)f\|_p : f \in A_{p,j}, \|f\|_p \leq 1 \right\} \leq c_1 \cdot c_{\text{interp}}(p) \cdot 2^{(j'-j)s}. \end{aligned}$$

Proof. Suppose first that $p = \infty$. Let $j_1, j'_1 \in \mathbb{Z}$ and $j'_1 \leq j_1$. Let $g \in A_{\infty, j'_1}$ and

$$g(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^n} \mathbf{b}[\mathbf{k}] v_{j'_1, \mathbf{k}}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x}_0 \in \mathbb{R}^n$. Let $J := \{\mathbf{k} \in \mathbb{Z}^n : w_{j_1, \mathbf{k}}(\mathbf{x}_0)\}$ and $S_1 := \overline{B}_{\mathbb{R}^n}(\mathbf{x}_0; 2^{-j_1}(\rho_2 + \rho_3))$. Now $\#J \leq N_{\text{cover}}(w)$ and $M_{j_1, \mathbf{k}} \subset S_1$ for all $\mathbf{k} \in J$. Suppose that $s \in \mathbb{R}_+ \setminus \mathbb{Z}_+$. Let $m := \lfloor s \rfloor$ and $a := s - m$. Let t be the m th degree Taylor polynomial of g at \mathbf{x}_0 and $r := g - t$. It follows from the polynomial span of (\tilde{w}, w) that $(I - P_{j_1})g = r - P_{j_1}r$. Hence $((I - P_{j_1})g)(\mathbf{x}_0) = r(\mathbf{x}_0) - (P_{j_1}r)(\mathbf{x}_0) = -(P_{j_1}r)(\mathbf{x}_0)$. Let $\ell \in J$ and $\mathbf{y} \in M_{j_1, \ell}$. Now $\|\mathbf{y} - \mathbf{x}_0\| \leq 2^{-j_1}(\rho_2 + \rho_3)$. If $m = 0$ we have $|r(\mathbf{y})| = |g(\mathbf{y}) - g(\mathbf{x}_0)| \leq H(g; 0, a)\|\mathbf{y} - \mathbf{x}_0\|^s \leq c_2 H(g; 0, a) \cdot 2^{-js}$. If $m > 0$ it follows from Lemma 10.30 that $|r(\mathbf{y})| \leq c_3 H(g; m, a)\|\mathbf{y} - \mathbf{x}_0\|^s \leq c_4 H(g; m, a) \cdot 2^{-js}$. By Lemma 10.29 we have $H(g; m, a) \leq c_5 \cdot 2^{j's} H(v; m, a)\|\mathbf{b}\|_{\infty}$. We also have $\|\mathbf{b}\|_{\infty} \leq c_6 \|g\|_{\infty}$ where c_6 does not depend on \mathbf{b} or g . Thus $|\langle \tilde{z}_{j_1, \ell}, r|_{M_{j_1, \ell}} \rangle| \leq c_8 H(v; m, a) \cdot 2^{(j'_1 - j_1)s} \|\mathbf{b}\|_{\infty} \leq c_9 \cdot 2^{(j'_1 - j_1)s} \|g\|_{\infty}$. Hence

$$|((I - P_{j_1})g)(\mathbf{x}_0)| \leq \sum_{\mathbf{k} \in J} |\langle \tilde{z}_{j_1, \mathbf{k}}, r|_{M_{j_1, \mathbf{k}}} \rangle| |w_{j_1, \mathbf{k}}(\mathbf{x}_0)| \leq N_{\text{cover}}(w) \|w\|_{\infty} \cdot c_9 \cdot 2^{(j'_1 - j_1)s} \|g\|_{\infty}.$$

Suppose then that $s \in \mathbb{Z}_+$. Let t be the $m - 1$ degree Taylor polynomial of g at point \mathbf{x}_0 and $r := g - t$. The result follows from

$$r(\mathbf{y}) = \frac{1}{m!} \sum_{i_1, \dots, i_m=1}^n f_{i_1, \dots, i_m}(\mathbf{x}_0 + c(\mathbf{y})(\mathbf{y} - \mathbf{x}_0)) \prod_{l=1}^m (\mathbf{y}[i_l] - \mathbf{x}_0[i_l]).$$

We have

$$\text{supp}(I - P_j)v_{j', \mathbf{k}} \subset \overline{B}_{\mathbb{R}^n}(2^{-j'}\mathbf{k}; 2^{-j}(\rho_2 + \rho_3) + 2^{-j'}\rho_1) \quad (69)$$

for all $j, j' \in \mathbb{Z}$ and $j' \leq j$. The result in case $p = 1$ follows from Equation (69), Lemma 10.23 and from case $p = \infty$.

When $p \in]1, \infty[$ the result follows from Lemma 10.22 and Banach space interpolation. \square

Lemma 10.32. *Let $\sigma, s \in \mathbb{R}_+$, $s > \sigma$, $p \in [1, \infty]$, and $q \in [1, \infty]$. Suppose that $\varphi^{[n]} \in C^s(\mathbb{R}^n)$. Then $\varphi^{[n]} \in B_{p, q}^{\sigma}(\mathbb{R}^n)$.*

Lemma 10.33. *Let $p \in [1, \infty]$, $q \in [1, \infty]$, $\sigma \in \mathbb{R}_+$, and $n/p < \sigma < r_0$. Let $\varphi^{[n]}$ be a scaling function of a compactly supported interpolating tensor product MRA of $C_u(\mathbb{R}^n)$ and $\bar{\varphi} : \mathbb{R}^n \rightarrow K$ a compactly supported and continuous mother scaling function of a $(\lfloor \sigma \rfloor + 1)$ -regular orthonormal wavelet family. Suppose that $\varphi^{[n]} \in C^{r_0}(\mathbb{R}^n)$ and $(\tilde{\varphi}^{[n]}, \varphi^{[n]})$ spans all the polynomials of degree at most $\lfloor \sigma \rfloor - 1$. Then $\|\cdot\|_{B_{p, q}^{\sigma}(\mathbb{R}^n); j_0}^{(i)}$ and $\|\cdot\|_{B_{p, q}^{\sigma}(\mathbb{R}^n); j_0}^{(o)}$ are equivalent norms on the vector space $B_{p, q}^{\sigma}(\mathbb{R}^n) \cap C_u(\mathbb{R}^n)$.*

Proof. We have $\bar{\varphi} \in C^{\bar{r}_0}(\mathbb{R}^n)$ where $\sigma < \bar{r}_0 < \lfloor \sigma \rfloor + 1$. By [34, section 2.6 corollary] $(2^{nj}\bar{\varphi}^*, \bar{\varphi})$ spans all the polynomials of degree at most $\lfloor \sigma \rfloor + 1$. The proof is based on Lemmas 3.22, 10.16, 10.17, 10.27, 10.28, and 10.31 and Equation (68). \square

Lemma 10.34. *Let $p \in [1, \infty]$, $q \in [1, \infty]$, $\sigma \in \mathbb{R}_+$, $j_0 \in \mathbb{Z}$, $r_0 \in \mathbb{R}_+$, and $n/p < \sigma < r_0$. Let $\varphi^{[n]}$ be a scaling function of a compactly supported interpolating tensor product MRA of $C_u(\mathbb{R}^n)$ and suppose that $\varphi^{[n]} \in B_{p, q}^{\sigma}(\mathbb{R}^n)$ and $\varphi^{[n]} \in C^{r_0}(\mathbb{R}^n)$. Then $\|\cdot\|_{B_{p, q}^{\sigma}(\mathbb{R}^n); j_0}^{(i)}$ and $\|\cdot\|_{B_{p, q}^{\sigma}(\mathbb{R}^n); j_0}^{(w)}$ are equivalent norms on the vector space $B_{p, q}^{\sigma}(\mathbb{R}^n) \cap C_u(\mathbb{R}^n)$.*

Proof. Use Lemmas 10.24 and 10.26. \square

Theorem 10.35. Let $p \in [1, \infty]$, $q \in [1, \infty]$, $\sigma \in]0, 1[$, $r_0 \in \mathbb{R}_+$, and $n/p < \sigma < r_0$. Let $\varphi^{[n]} : \mathbb{R}^n \rightarrow K$ be a scaling function of a compactly supported interpolating tensor product MRA of $C_u(\mathbb{R}^n)$ and suppose that $\varphi^{[n]} \in C^{r_0}(\mathbb{R}^n)$. Suppose also that $(\tilde{\varphi}^{[n]}, \varphi^{[n]})$ spans all the polynomials of degree at most $\lceil \sigma \rceil - 1$. Then $\|\cdot\|_{B_{p,q}^\sigma(\mathbb{R}^n); j_0}^{(w)}$ and $\|\cdot\|_{B_{p,q}^\sigma(\mathbb{R}^n); j_0}^{(i)}$ are equivalent to the restriction of some norm of Besov space $B_{p,q}^\sigma(\mathbb{R}^n)$ onto the vector space $B_{p,q}^\sigma(\mathbb{R}^n) \cap C_u(\mathbb{R}^n)$.

Proof. The theorem follows from Lemmas 10.33 and 10.34. \square

10.3 Consequences of the Besov Space Norm Equivalence

In particular, the Besov space norm equivalence holds for Hölder spaces $C^\sigma(\mathbb{R}^n)$ for $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. When $n \in \mathbb{Z}_+$, $\sigma \in \mathbb{R}_+$, and $q \in [1, \infty]$ we have $B_{\infty,q}^\sigma(\mathbb{R}^n) \subset_{\text{set}} C_u(\mathbb{R}^n)$ [41, prop. 2.3.2/2(i) and eq. (2.3.5/1)].

Definition 10.36. When $n \in \mathbb{Z}_+$ and $j_0 \in \mathbb{Z}$ define

$$\Omega(n, j_0) := \{(\mathbf{0}_n, j_0, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\} \cup \bigcup_{j=j_0}^{\infty} \bigcup_{\mathbf{s} \in J_+(n)} \{(\mathbf{s}, j, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}.$$

Theorem 10.37. Let $n \in \mathbb{Z}_+$, $\sigma \in \mathbb{R}_+$, and $j_0 \in \mathbb{Z}$. Let $\varphi^{[n]}$ be a mother scaling function of a compactly supported tensor product MRA of $C_u(\mathbb{R}^n)$. Suppose that $\varphi^{[n]} \in C^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}_+$, $r > \sigma$. Then the sequence $(\psi_{\iota(k)})_{k=0}^{\infty}$ is not a Schauder basis of $B_{\infty,\infty}^\sigma(\mathbb{R}^n) =_{\text{tvs}} \mathcal{Z}^\sigma(\mathbb{R}^n)$ with any summing order ι where ι is a bijection from \mathbb{N} onto $\Omega(n, j_0)$.

Proof. Define $j(\mathbf{a}) = j'$ for all $\mathbf{a} = (\mathbf{s}', j', \mathbf{k}')$, $\mathbf{s}' \in \{0, 1\}^n$, $j' \in \mathbb{Z}$, and $\mathbf{k}' \in \mathbb{Z}^n$. There exists $r_1 \in \mathbb{Z}_+$ so that $\text{supp } \varphi^{[n]} \in \overline{B}_{\mathbb{R}^n}(0; r_1)$. Let $c := 2r_1 + \lceil 2^{j_0} \rceil (2\lceil \sigma \rceil + 1)$ and $m := \lceil r \rceil$. Let

$$\begin{aligned} \eta(k) &:= c \cdot 2^k k \mathbf{e}_1^{[n]} \\ a_k &:= \psi_{\mathbf{e}_1^{[n]}, j_0+k, \eta(k)}^{[n]} \\ g_k &:= 2^{-(j_0+k)\sigma} a_k \\ z &:= \psi_{\mathbf{e}_1^{[n]}}^{[n]} \end{aligned}$$

where $k \in \mathbb{N}$ and

$$f_l := \sum_{k=0}^l g_k$$

where $l \in \mathbb{N}$. Let f be the limit of sequence $(f_l)_{l=0}^{\infty}$ in $C_0(\mathbb{R}^n)$. We have

$$\sum_{k=0}^{\infty} \|g_k\|_{\infty} = \|z\|_{\infty} \sum_{k=0}^{\infty} 2^{-(j_0+k)s} \in \mathbb{R}_+$$

and hence series $\sum_{k=0}^{\infty} g_k$ converges absolutely in Banach space $C_0(\mathbb{R}^n)$. We also have $a_k, g_k \in B_{\infty,\infty}^\sigma(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. Define $S_{j,\ell}(y) := \overline{B}_{\mathbb{R}^n}(2^{-j}\ell; 2^{-j}r_1 + y)$ where $j \in \mathbb{Z}$, $\ell \in \mathbb{Z}^n$, and $y \in \mathbb{R}_0$. Define $\alpha(k) := (\beta(k), \eta(k))$ where $\beta(k) := j_0 + k$. Suppose that $t_1 \in]0, 1[$. We have

$$\text{supp}_{\text{set}} \Delta_{\mathbf{h}}^m g_k = \text{supp}_{\text{set}} \Delta_{\mathbf{h}}^m a_k \subset S_{\beta(k), \eta(k)}(m) \quad (70)$$

for all $\mathbf{h} \in \overline{B}_{\mathbb{R}^n}(0; t_1)$ and $k \in \mathbb{N}$. Furthermore,

$$S_{\beta(k), \eta(k)}(m) \cap S_{\beta(l), \eta(l)}(m) = \emptyset \quad (71)$$

for all $k, l \in \mathbb{N}$ and $k \neq l$. Using Equations (70) and (71) we obtain

$$\omega_\infty^m(f; t_1) \leq \sup\{\omega_\infty^m(g_k; t_1) : k \in \mathbb{N}\}. \quad (72)$$

Furthermore, $\omega_\infty^m(a_k; t_1) = \omega_\infty^m(z; 2^{\beta(k)}t_1)$ for all $k \in \mathbb{N}$. Consequently $\omega_\infty^m(z; 2^{\beta(k)}t_1) \leq c_2 \cdot (2^{\beta(k)}t_1)^\sigma$ where $c_2 := \|f\|_{B_{\infty,\infty}^{\sigma}(\mathbb{R}^n)}^{(c')}$. Furthermore, $\omega_\infty^m(g_k; t_1) = 2^{-(j_0+k)\sigma} \omega_\infty^m(a_k; t_1) \leq c_2 t_1^\sigma$. By Equation (72) we have $\omega_\infty^m(f; t_1) \leq c_2 t_1^\sigma$. Thus $f \in B_{\infty,\infty}^\sigma(\mathbb{R}^n)$.

Suppose that $(\psi_{\iota(k)})_{j=j_0}^\infty$ would be a Schauder basis of $B_{\infty,\infty}^\sigma(\mathbb{R}^n)$. Then

$$f = \sum_{k=0}^\infty \mathbf{d}[\iota(k)] \psi_{\iota(k)}$$

for some sequence $\mathbf{d} \in \mathbb{C}^{\Omega(n,j_0)}$. Define

$$\xi_\beta := \sum_{k=0}^\beta \mathbf{d}[\iota(k)] \psi_{\iota(k)}$$

where $\beta \in \mathbb{N}$. Let $b \in \mathbb{N}$. Define

$$j_1 := \max\{j(\alpha(k)) \in \mathbb{Z} : k \in Z_0(b)\} + 1.$$

There exists $k_1 \in \mathbb{N}$, $k_1 > b$ so that $\iota(k_1) = \alpha(j_1 - j_0)$. Let $k_2 := j_1 - j_0$. Now

$$\iota(k_1) = (\mathbf{e}_1^{[n]}, j_1, c \cdot 2^{k_2} k_2 \mathbf{e}_1^{[n]}).$$

We have

$$\left| \langle \tilde{\psi}_{\iota(k_1)}, f - \xi_b \rangle \right| \leq \left\| \left(\langle \tilde{\psi}_{\mathbf{s}, j_1, \mathbf{k}}^{[n]}, f - \xi_b \rangle \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_\infty$$

and $\left| \langle \tilde{\psi}_{\iota(k_1)}^{[n]}, f - \xi_b \rangle \right| = 2^{-(j_0+k_2)\sigma}$. Hence

$$\begin{aligned} 1 &\leq 2^{(j_0+k_2)\sigma} \left\| \left(\langle \tilde{\psi}_{\mathbf{s}, j_1, \mathbf{k}}^{[n]}, f - \xi_b \rangle \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_\infty \\ &\leq \left\| \left(2^{j\sigma} \left\| \left(\langle \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}, f - \xi_b \rangle \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_\infty \right)_{j=j_0}^\infty \right\|_\infty \\ &\leq \|f - \xi_b\|_{B_{\infty,\infty}^\sigma(\mathbb{R}^n); j_0}^{(w)}. \end{aligned}$$

Thus $\xi_\beta \not\rightarrow f$ in Banach space $B_{\infty,\infty}^\sigma(\mathbb{R}^n)$ as $\beta \rightarrow \infty$. Consequently $(\psi_{\iota(k)})_{j=j_0}^\infty$ is not a Schauder basis of $B_{\infty,\infty}^\sigma(\mathbb{R}^n)$. \square

Theorem 10.38. *Let $n \in \mathbb{Z}_+$, $\sigma \in \mathbb{R}_+$, $q \in [1, \infty[$, and $j_0 \in \mathbb{Z}$. Let $\varphi^{[n]}$ be a mother scaling function of a compactly supported tensor product MRA of $C_0(\mathbb{R}^n, K)$. Suppose that $\varphi^{[n]} \in C^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}_+$, $r > \sigma$. Then $\{\psi_\alpha : \alpha \in \Omega(n, j_0)\}$ is an unconditional basis of Banach space $B_{\infty,q}^\sigma(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ equipped with a norm of $B_{\infty,q}^\sigma(\mathbb{R}^n)$ and the coefficient functional corresponding to basis vector ψ_α is $\tilde{\psi}_\alpha$ for each $\alpha \in \Omega(n, j_0)$.*

Proof. Let

$$\begin{aligned} S_j(m) &:= \{(\mathbf{0}_n, j, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n, \|\mathbf{k}\|_2 \leq m\}, \quad j \in \mathbb{Z}, m \in \mathbb{N} \\ D_{\mathbf{s},j}(m) &:= \{(\mathbf{s}, j, \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n, \|\mathbf{k}\|_2 \leq m\}, \quad \mathbf{s} \in \{0, 1\}^n, j \in \mathbb{Z}, m \in \mathbb{N} \\ D_j(m) &:= \bigcup_{\mathbf{s} \in J_+(n)} D_{\mathbf{s},j}(m), \quad j \in \mathbb{Z}, m \in \mathbb{N}. \end{aligned}$$

Let

$$\begin{aligned} m_1 &:= \min\{m \in \mathbb{N} : \forall \mathbf{k} \in \mathbb{Z}^n : (\|\mathbf{k}\|_\infty > m \implies \forall \mathbf{s} \in \{0, 1\}^n : \tilde{g}_{\mathbf{s}, \mathbf{k}}^{[n]} = 0)\} \\ m_2 &:= \lceil m_1 \sqrt{n} \rceil \\ A(k) &:= S_{j_0}(2^{j_0}k) \cup \bigcup_{l=j_0}^{j_0+k-1} D_l(m_2 + 2^{j_0+k}k), \quad k \in \mathbb{Z}_+. \end{aligned}$$

Define $\mathbf{s}(\mathbf{a}) = \mathbf{s}'$, $j(\mathbf{a}) = j'$, and $\mathbf{k}(\mathbf{a}) = \mathbf{k}'$ for all $\mathbf{a} = (\mathbf{s}', j', \mathbf{k}')$, $\mathbf{s}' \in \{0, 1\}^n$, $j' \in \mathbb{Z}$, and $\mathbf{k}' \in \mathbb{Z}^n$. Let $f \in B_{\infty, q}^\sigma(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. Let $\eta : \mathbb{N} \rightarrow \Omega(n, j_0)$ be a bijection. Let $\beta_\alpha := \langle \tilde{\psi}_\alpha, f \rangle$ for all $\alpha \in \Omega(n, j_0)$. Define

$$\beta_\alpha^{(m)} := \begin{cases} \langle \tilde{\psi}_\alpha, f \rangle; & \alpha \notin \eta[Z_0(m)] \\ 0; & \text{otherwise} \end{cases}$$

for all $m \in \mathbb{N}$ and $\alpha \in \Omega(n, j_0)$. Define

$$\xi_m := \sum_{\alpha \in \Omega(n, j_0)} \beta_\alpha^{(m)} \psi_\alpha$$

for all $m \in \mathbb{N}$. There exists $c_1 \in \mathbb{R}_+$ so that

$$\left| \langle \tilde{\psi}_\alpha, f \rangle \right| \leq c_1 \|f\|_\infty \quad (73)$$

for all $f \in C_0(\mathbb{R}^n)$ and $\alpha \in \Omega(n, j_0)$.

Let $h \in \mathbb{R}_+$. Choose $j_1 \in \mathbb{Z}$, $j_1 > \max\{j_0, 0\}$ so that

$$\left\| \left(2^{j\sigma} \left\| \left(\langle \tilde{\psi}_{\mathbf{s}, j, \mathbf{k}}^{[n]}, f \rangle \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_\infty \right)_{j=j_1}^\infty \right\|_q < \frac{h}{4}. \quad (74)$$

Choose $r_1 \in \mathbb{R}_+$ so that

$$\sup\{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\| \geq r_1\} < \frac{h}{2^{n+2+j_1\sigma}(j_1 - j_0)c_1} \quad (75)$$

and let $m_3 := \max\{j_1 - j_0, \lceil r_1 \rceil\}$.

Choose $m_4 \in \mathbb{Z}_+$ so that $A(m_3) \subset \eta[Z_0(m_4)]$. Suppose that $j_2 \in \mathbb{Z}$, $j_0 \leq j_2 < j_1$. Let $l \in \mathbb{Z}$, $l > m_4$, and $j(\eta(l)) = j_2$. If $\mathbf{s}(\eta(l)) = \mathbf{0}_n$ we have $j(\eta(l)) = j_0$ and $\|\mathbf{k}(\eta(l))\|_2 > 2^{j_0}m_3 \geq 2^{j_0}r_1$ from which it follows that

$$\left| \langle \tilde{\psi}_{\eta(l)}, f \rangle \right| = \left| f\left(\frac{\mathbf{k}(\eta(l))}{2^{j_0}}\right) \right| < \frac{h}{4}.$$

Consequently

$$\left\| (\beta_{\mathbf{0}_n, j_0, \mathbf{k}}^{(m_4)})_{\mathbf{k} \in \mathbb{Z}^n} \right\|_\infty < \frac{h}{4}. \quad (76)$$

Suppose then that $\mathbf{s}(\eta(l)) \neq \mathbf{0}_n$. Now $\|\mathbf{k}(\eta(l))\|_2 > m_2 + 2^{j_0+m_3}m_3$ from which it follows that $\forall \ell \in \mathbb{Z}^n : \|\ell\|_\infty \leq m_1 \implies \|\mathbf{k}(\eta(l)) + \ell\|_2 \geq 2^{j_0+m_3}m_3$. We have $2^{j_0+m_3} > 2^{j_2}$ and $\forall \ell \in \mathbb{Z}^n : \|\ell\|_\infty \leq m_1 \implies \|\mathbf{k}(\eta(l)) + \ell\|_2 > 2^{j_2}m_3 \geq 2^{j_2}r_1$. By Equations (73) and (75) we have

$$\left| \langle \tilde{\psi}_{\eta(l)}, f \rangle \right| < \frac{h}{2^{n+2+j_1\sigma}(j_1 - j_0)}.$$

Hence

$$\left\| \left(\beta_{\mathbf{s}(\eta(l)), j_2, \mathbf{k}}^{(m_4)} \right)_{\mathbf{k} \in \mathbb{Z}^n} \right\|_\infty < \frac{h}{2^{n+2+j_1\sigma}(j_1 - j_0)}.$$

from which it follows that

$$2^{j_2\sigma} \left\| \left(\beta_{\mathbf{s}(\eta(l)), j_2, \mathbf{k}}^{(m_4)} \right)_{\mathbf{k} \in \mathbb{Z}^n} \right\|_{\infty} < \frac{h}{2^{n+2}(j_1 - j_0)}.$$

Consequently

$$\begin{aligned} \left\| \left(2^{j\sigma} \left\| \left(\beta_{\mathbf{s}, j, \mathbf{k}}^{(m_4)} \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_{\infty} \right)_{j=j_0}^{j_1-1} \right\|_{l^q(j_0 + \mathbb{N})} &< \left(\left(\frac{h}{2^{n+2}(j_1 - j_0)} \right)^q (j_1 - j_0) \right)^{\frac{1}{q}} \\ &< \frac{1}{4}h. \end{aligned} \quad (77)$$

Thus by Equations (74) and (77)

$$\begin{aligned} &\left\| \left(2^{j\sigma} \left\| \left(\beta_{\mathbf{s}, j, \mathbf{k}}^{(m_4)} \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_{\infty} \right)_{j=j_0}^{\infty} \right\|_{l^q(j_0 + \mathbb{N})}^q \\ &= \left\| \left(2^{j\sigma} \left\| \left(\beta_{\mathbf{s}, j, \mathbf{k}}^{(m_4)} \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_{\infty} \right)_{j=j_0}^{j_1-1} \right\|_{l^q(\{j_0, \dots, j_1-1\})}^q \\ &\quad + \left\| \left(2^{j\sigma} \left\| \left(\beta_{\mathbf{s}, j, \mathbf{k}}^{(m_4)} \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_{\infty} \right)_{j=j_1}^{\infty} \right\|_{l^q(j_1 + \mathbb{N})}^q \\ &< \left(\frac{h}{4} \right)^q + \left(\frac{h}{4} \right)^q = \frac{2}{4^q} h^q. \end{aligned}$$

It follows that

$$\left\| \left(2^{j\sigma} \left\| \left(\beta_{\mathbf{s}, j, \mathbf{k}}^{(m_4)} \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_{\infty} \right)_{j=j_0}^{\infty} \right\|_{l^q(j_0 + \mathbb{N})} < \frac{h}{2}. \quad (78)$$

By Equations (76) and (78) we have $\|\xi_{m_4}\|_{B_{\infty, q}^{\sigma}(\mathbb{R}^n); j_0}^{(w)} < h$. Hence $\|\xi_m\|_{B_{\infty, q}^{\sigma}(\mathbb{R}^n); j_0}^{(w)} \rightarrow 0$ as $m \rightarrow \infty$. It follows that

$$\left\| \sum_{l=0}^{\infty} \beta_{\eta(l)}^{(m)} \psi_{\eta(l)} \right\|_{B_{\infty, q}^{\sigma}(\mathbb{R}^n)} = \left\| \sum_{l=m+1}^{\infty} \beta_{\eta(l)} \psi_{\eta(l)} \right\|_{B_{\infty, q}^{\sigma}(\mathbb{R}^n)} \rightarrow 0$$

as $m \rightarrow \infty$. Using Definition 2.18 we get $\|\xi_m|_{C_0(\mathbb{R}^n)}\| \rightarrow 0$ as $m \rightarrow \infty$. We also have

$$s_m := \sum_{l=0}^m \beta_{\eta(l)} \psi_{\eta(l)} \rightarrow g \in B_{\infty, q}^{\sigma}(\mathbb{R}^n)$$

from which it follows that $\|g - s_m\|_{B_{\infty, q}^{\sigma}(\mathbb{R}^n)} \rightarrow 0$ (use some equivalent norm of the Besov space) as $m \rightarrow \infty$. It follows from Definition 2.18 that $\|g - s_m\|_{C_0(\mathbb{R}^n)} \rightarrow 0$ as $m \rightarrow \infty$. Hence $g \in C_0(\mathbb{R}^n)$ and $s_m \rightarrow g$ in $C_0(\mathbb{R}^n)$ as $m \rightarrow \infty$. If we had $g \neq f$ we would have $\langle \tilde{\psi}_{\gamma}, f - g \rangle = \langle \tilde{\psi}_{\gamma}, f \rangle - \langle \tilde{\psi}_{\gamma}, g \rangle \neq 0$ for some $\gamma \in \Omega(n, j_0)$. Now $\gamma = \eta(l_0)$ for some $l_0 \in \mathbb{N}$ and $\langle \tilde{\psi}_{\eta(l_0)}, g \rangle = \beta_{\eta(l_0)} = \langle \tilde{\psi}_{\eta(l_0)}, f \rangle$, which is a contradiction. Hence $f = g$. \square

Theorem 10.39. *Let $n \in \mathbb{Z}_+$ and $j_0 \in \mathbb{Z}$. Let $\sigma \in \mathbb{R}_+$, $p \in [1, \infty]$, and $q \in [1, \infty]$. Let $\varphi^{[n]}$ be a mother scaling function of a compactly supported tensor product MRA of $C_u(\mathbb{R}^n)$. Suppose that $\varphi^{[n]} \in C^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}_+$, $r > \sigma$. Let $f \in B_{\infty, q}^{\sigma}(\mathbb{R}^n)$. Then*

$$f(\mathbf{x}) = \sum_{\alpha \in \Omega(n, j_0)} \langle \tilde{\psi}_{\alpha}, f \rangle \psi_{\alpha}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and the series above converges absolutely for each $\mathbf{x} \in \mathbb{R}^n$.

Proof. Let

$$a := \left\| \left(2^{(\sigma - \frac{n}{p})j} \left\| \left(\langle \tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]}, f \rangle \right)_{\mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n} \right\|_p \right)_{j=j_0}^\infty \right\|_q.$$

Now

$$\left| \langle \tilde{\psi}_{\mathbf{s},j,\mathbf{k}}^{[n]}, f \rangle \right| \leq 2^{-(\sigma - \frac{n}{p})j} a \quad (79)$$

for all $\mathbf{s} \in J_+(n)$, $j \in \mathbb{N} + j_0$, and $\mathbf{k} \in \mathbb{Z}^n$. Let

$$A_0(\mathbf{x}) := \left\{ (\mathbf{0}_n, j_0, \mathbf{k}) : \mathbf{x} \in \text{supp } \varphi_{j_0, \mathbf{k}}^{[n]}, \mathbf{k} \in \mathbb{Z}^n \right\}$$

for each $\mathbf{x} \in \mathbb{R}^n$ and

$$A_k(\mathbf{x}) := \left\{ (\mathbf{s}, j_0 + k - 1, \mathbf{k}) : \mathbf{x} \in \text{supp } \psi_{\mathbf{s}, j_0 + k - 1, \mathbf{k}}^{[n]}, \mathbf{s} \in J_+(n), \mathbf{k} \in \mathbb{Z}^n \right\}$$

for each $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$. Let $\eta : \mathbb{N} \rightarrow \Omega(n, j_0)$ be a bijection and define

$$\beta_\alpha^{(m)} := \begin{cases} \langle \tilde{\psi}_\alpha, f \rangle; & \alpha \notin \eta[Z_0(m)] \\ 0; & \alpha \in \eta[Z_0(m)] \end{cases} \quad (80)$$

for each $m \in \mathbb{N}$ and $\alpha \in \Omega(n, j_0)$. Let

$$g_m(\mathbf{x}) := \sum_{k=0}^m \langle \tilde{\psi}_{\eta(k)}, f \rangle \psi_{\eta(k)}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$ and $m \in \mathbb{N}$. It follows from Theorem 6.16 that

$$f(\mathbf{x}) = \sum_{k=0}^\infty \sum_{\alpha \in A_k(\mathbf{x})} \langle \tilde{\psi}_\alpha, f \rangle \psi_\alpha(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^n$. Define $c_1 := \max\{\|\psi_{\mathbf{s}}^{[n]}\|_\infty : \mathbf{s} \in \{0, 1\}^n\}$. Let $\mathbf{y} \in \mathbb{R}^n$ and $h \in \mathbb{R}_+$. Define $m_1 := \max\{\#A_k(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$. Now $m_1 \in \mathbb{Z}_+$. Let

$$j_1 := \max \left\{ j_0 + 1, \left\lceil \left(\sigma - \frac{n}{p} \right)^{-1} \log_2 \frac{c_1 m_1 a}{(1 - 2^{-(\sigma - \frac{n}{p})h})} \right\rceil \right\}$$

Now

$$2^{-(\sigma - \frac{n}{p})j_1} a < \frac{(1 - 2^{-(\sigma - \frac{n}{p})h})}{c_1 m_1}.$$

Choose $m_2 \in \mathbb{N}$ so that $A_k(\mathbf{y}) \subset \eta[Z_0(m_2)]$ for all $k \in Z_0(j_1 - j_0)$. Suppose that $m \in \mathbb{N}$, $m > m_2$, and $\alpha_0 \in A_{j_0+k}$ for some $k_0 \in Z_0(j_1 - j_0)$. Now $\alpha_0 \in \eta[Z_0(m_2)] \subset \eta[Z_0(m)]$. By Equation (80) we have $\beta_{\alpha_0}^{(m)} = 0$. Hence by Equation (79)

$$\begin{aligned} |f(\mathbf{y}) - g_m(\mathbf{y})| &\leq c_1 \sum_{k=0}^\infty \sum_{\alpha \in A_k(\mathbf{y})} |\beta_\alpha^{(m)}| = c_1 \sum_{k=j_1-j_0+1}^\infty \sum_{\alpha \in A_k(\mathbf{y})} |\beta_\alpha^{(m)}| \\ &\leq c_1 \sum_{k=j_1-j_0+1}^\infty m_1 \cdot 2^{-(j_0+k-1)(\sigma - \frac{n}{p})} \cdot a \\ &= c_1 m_1 a \cdot 2^{-j_1(\sigma - \frac{n}{p})} \frac{1}{1 - 2^{-(\sigma - \frac{n}{p})}} \\ &< h \end{aligned}$$

Thus $g_m(\mathbf{x}) \rightarrow f(\mathbf{x})$ as $m \rightarrow \infty$ for all $\mathbf{x} \in \mathbb{R}^n$. □

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